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Elliptic theta functions and the fractional quantum Hall effect

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Abstract

Algebro-geometric methods are applied to the theoretical understanding of the fractionary quantum Hall effect on a periodic lattice. The fermionic Fock space of the many-electron system is precisely identified, and as a consequence, the variational Haldane–Rezayi ground state is decomposed in terms of one-particle wave functions at the first Landau level; the filling factor is thus analytically computed. Quasi-hole and quasi-particle excitations are also analyzed. The center of mass dynamics is described in terms of a section in a very subtle stable vector bundle. The Hall conductance arises as a topological invariant; namely, the slope of the vector bundle previously mentioned. © 1998 Elsevier Science B.V.

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1. Introduction

During the century following its discovery, the classical Hall effect [1] has been proved to be very effective in determining the electronic properties of many conducting systems in solid state and condensed matter physics. Electric currents $\boldsymbol{j}$ in the presence of constant magnetic fields $\boldsymbol{B}$ perpendicular to $\boldsymbol{j}$ are deflected by the Lorentz force; as a consequence

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an electric field $\boldsymbol{E}$ proportional to $\boldsymbol{j} \wedge \boldsymbol{B}$ builds up. The conductivity tensor $\boldsymbol{j} = \sigma \boldsymbol{E}$ is off-diagonal with non-zero components of the form $\pm\sigma_{\mathrm{H}}\delta$. σ_{H} is usually referred to as Hall conductance; δ is the thickness of the sample, and it happens that in the classical Hall experiment $\sigma_{\mathrm{H}} = (\mathrm{e}^2/h)(\nu_{\mathrm{H}})$. The filling factor ν_{H} is the ratio of the number (density) of electrons to the number (density) of Landau states, organized in degenerated levels, in the sample.

Circa 1980 spectacular experiments by von Klitzig et al. and Tsui et al. (see, e.g., [2]) unveiled a completely new scenario: in two-dimensional electronic systems, such as occur in MOSFET devices and/or semi-conductor heterojunctions, the dependence of σ_{H} on ν_{H} departs drastically from the classical behavior if the experiment is performed at very low temperatures and very high magnetic fields. The new features are as follows:

1. σ_{H} is always $n\mathrm{e}^2/h$, in MOSFET devices, and/or $(n/m)\mathrm{e}^2/h$, in heterojunctions, n, m being integers. The Hall conductance is quantized and this phenomenon is referred to as the quantum Hall effect, either integer, IQHE, or fractionary, FQHE.
2. There are plateaux around the special values $\nu_{\mathrm{H}} = n$ or $\nu_{\mathrm{H}} = n/m$ where the experimental outcome coincides with the classical case; varying ν_{H}, however, σ_{H} does not change in the QHE.

Under the circumstances of the QHE σ_{H} is independent of the geometrical and internal structures of the Hall devices, as well as of the magnetic field $\boldsymbol{B}$, to an extraordinary high level of precision. The quantum Hall effect thus becomes a universal phenomenon with a deep origin and very rich structures. In the last 15 years we have witnessed how theoretical explanations have developed rapidly and how experimental aspects have been exhaustively analyzed. An excellent summary may be found in [2].

A satisfactory theoretical explanation of both the fractionality of σ_{H} and the existence of plateaux in the FQHE – points 1 and 2 above – has not yet been completely developed. The mechanism giving rise to the plateaux is quite unclear – see [3] – as opposed to our understanding of the same phenomenon in the IQHE, where it seems to be due to the existence of localized states; this kind of state does not exist in high-mobility electron systems. Rather than impurities, electron–electron interactions play an important role in the FQHE and the crucial issue is the search for ground states of the many-electron system giving fractional Hall conductivity. Laughlin theory [4] rests on a variational wave function of Jastrow-type which succesfully describes the ground state of a two-dimensional gas of interacting electrons under the circumstances of the FQHE as a new kind of quantum fluid; the associated filling factor is $1/m$, with m odd, and thus solves point 1.

With respect to point 2, the existence of a gap in σ_{H} calls for a topological interpretation of the Hall conductivity in the same spirit as the description of Thouless [5] of σ_{H} in the IQHE as the first Chern class of a line bundle over the torus parametrizing the first magnetic Brillouin zone. As mentioned above, the reasons why $\sigma_h = ne^2/h$, IQHE , and $\sigma_{\mathrm{H}} = (n/m)e^2/h$, FQHE, remain constant when the filling factor is slightly changed are different; thus, the Thouless approach without any modification cannot be applied to explain plateaux in the FQHE. Attempts to generalize the topological nature of σ_{H} to the fractionary

QHE start out from considering that the phenomenon occurs on a periodic lattice instead of on the complex plane or a compact subspace of it, see [6]. This leads to studying the problem on a torus, and for this and another reasons, e.g., numerical computations as in [7], the work on this subject in the physical literature is vast.

The main theme of this paper is to develop a precise mathematical description of the FQHE on a torus. The issues to be considered in the framework of the algebraic geometry throughout the work can be separated into two parts:

1.1. The Laughlin, quasi-hole and quasi-particle wave functions on a torus

From a mathematical point of view, the main idea to be developed concerning these variational states is the demonstration that the observations made by physicists about the ground state of the FQHE on a torus and its excitations are natural consequences of the classical addition formulas for elliptic Theta functions [8] and certain higher-order addition formulas that we prove.

Historically, it was Yoshioka, [9], who determined for the first time the fermionic Fock state built out of the first Landau level for a charged particle moving in a torus under a constant magnetic field. In [6], Tao and Haldane identified the first Landau level mentioned above as the space of sections of the line bundle of kth-*order* Theta functions over $\Sigma = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$: $\Gamma(\theta_\Sigma(ke))$, where Σ is the principal cell of the periodic lattice and τ is the modular parameter of the torus related to physical quantities, as we shall explain later. The main breakthrough, however, came from the work of Haldane and Rezayi [10]. In this seminal paper, the Laughlin wave function was generalized to the periodic case; as a consequence, the origin of the non-trivial center of mass dynamics of N electrons moving in a torus was understood. Moreover, variational wave functions for the elementary excitations of quasi-hole type on a torus were also proposed and a formal definition of quasi-particle wave functions was suggested by the authors.

In this context we address and clarify the following questions:

- The Haldane–Rezayi ground state mimicks the Laughlin wave function by replacing monomials in the relative coordinates $z_i - z_j$ of the ith and jth particles by odd Theta functions of the same arguments. (Magnetic) translation invariance of an N particle system in the plane calls for the profitable use of center of mass and relative coordinates. In order to understand the precise meaning of (magnetic) translationally invariant states of N particles on a torus, we must recall, e.g., for $N = 2$ particles, that the transformation $(z_1, z_2) \longrightarrow (z_1 + z_2, z_1 - z_2)$ *is not* a change of variables in the product $\Sigma \times \Sigma$ of the torus Σ by itself. This transformation is indeed a morphism $\Sigma \times \Sigma \xrightarrow{\xi} \Sigma \times \Sigma$ of degree 4, an observation which does not appear to have been recognized in the physics literature.
- Together with the higher-order addition formulas this inquiry allows us to prove the identity of the fermionic Fock space $\mathcal{F}(\Gamma(\theta_\Sigma(ke)))$ built out of the one-particle first Landau level with the space of sections of the line bundle where the Haldane–Rezayi state is defined. This provides the analog of the expansion of the Laughlin state as a

linear combination of Slater determinants and explicitly states that the filling factor for the Haldane–Rezayi wave function is $1/m = N/k$, as it should be. We thus find that the Haldane–Rezayi state is essentially unique and determined by a very broad principle of (magnetic) translation invariance restricted by the boundary conditions of the periodic lattice.

- We also identify the bundle where the quasi-hole wave function is defined as a holomorphic section and compute the quasi-hole fractionary charge in terms of a characteristic class of this bundle. We also show that the analog quasi-particle bundles do not admit global holomorphic sections so that quasi-particle wave functions on a torus cannot be defined inside the first Landau level.

1.2. The topological nature of the FQHE conductivity

Investigation of the dependence of the center of mass wave function of a system of N electrons in the FQHE phase on solenoid fluxes, one of which is inside a torus and the other passing through its center, led Tao and Haldane, see [6], to interpret Hall conductivity in terms of the first Chern class of a bundle where the Bloch magnetic functions are the sections. In a very interesting paper, Varnhagen [11] improved our knowledge of this conceptual advance by recognizing the bundle as a rank m vector bundle and suggested that the physically emerging plateaux correspond to stable bundles.

The second part of the paper deals with a rigorous mathematical formulation of Varnhagen's work and is devoted to:

(a) Identification of the vector bundle by means of a Fourier–Mukai transformation, see [12].
(b) Proof that the slope of the bundle coincides with the Hall conductivity which is therefore a topological invariant with rational values.
(c) Proof of the vector bundle stability when the denominator is odd.
(d) The demonstration of canonical isomorphisms between the vector bundles associated with systems of N and $N-1$ particles. This mathematically confirms physicists appriorístic point of view that only the center of mass dynamics matters with respect to the fractionality of σ_{H}.

Finally, our mathematical analysis suggests that one important role in the problem of unveiling the reason for the existence of plateaux in FQHE devices, with weak impurities and low levels of disorder, is played by the moduli space of stable bundles: changes in the filling factor of high-mobility electrons merely produce variations in the moduli space of stable bundles.

The organization of the paper is as follows: In Section 2 the Landau problem for a charged particle moving in a periodic lattice is solved and the first Landau level identified as spanned by a basis formed by elliptic Theta functions. Section 3 is devoted to the many-particle problem in increasing order of complexity. Quasi-holes and quasi-particles are also dealt with. In Section 4 an effective topological quantum field theory describing the center of mass dynamics is developed; the associated vector bundles are studied and the role of the slope is analyzed. Finally, in Section 5 a summary and outlook are offered.

2. The Landau problem in a periodic lattice

2.1. One-particle Hilbert space

If the motion takes place in a two-dimensional periodic lattice, the one-electron configuration space is the elliptic curve $\Sigma = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. Points in the complex plane $\mathbb{C}$ related by the discrete translation group $\Gamma : z \to z + n_1 + n_2\tau$, $n_1, n_2 \in \mathbb{Z}$, are identified. Here $z = x_1 + \mathrm{i}x_2$ is a non-dimensional local coordinate in $\mathbb{C}$: $x_1 = \tilde{x}_1/L_1$, $x_2 = \tilde{x}_2/L_1$, where L_1 is the length of the lattice in the direction chosen as the $\tilde{x}_1$-axis. The modular parameter $\tau = L_2 \mathrm{e}^{\mathrm{i}\theta}/L_1$ belongs to $\mathbf{H}$, the upper half-plane, because the whole system may be arranged in such a way that $\mathrm{Im}\tau > 0$, $L_2\mathrm{e}^{\mathrm{i}\theta}$ being the complex parameter of the second periodicity. We consider a line bundle $\mathcal{L}_k$ over Σ with connection and curvature given by

$$A = -\frac{2\pi k}{\mathrm{Im}\,\tau} x_2 \mathrm{d}x_1; \qquad F_A = \frac{2\pi k}{\mathrm{Im}\,\tau} \mathrm{d}x_1 \mathrm{d}x_2. \tag{1}$$

The degree of the bundle is $c_1(\mathcal{L}_k) = 1/2\pi \int_\Sigma F_A = k \in \mathbb{Z}$ and is related with the external constant magnetic field as follows:

$$2\pi k = \frac{eB}{hc} \frac{L_1 L_2 \sin\theta}{\mathrm{Im}\,\tau} \equiv \frac{eB}{hc} L_1^2. \tag{2}$$

The quantum space of states of this system is the space of L^2-integrable sections in $\mathcal{L}_k$, $\Gamma(\Sigma, \mathcal{L}_k)$. The quantum dynamics is essentially due to the Laplacian acting on $\Gamma(\Sigma, \mathcal{L}_k)$. The "creation" and "anihilation" operators, in physicist's terminology,

$$a = \sqrt{\frac{\mathrm{Im}\,\tau}{\pi k}} \left(\frac{\partial}{\partial \bar{z}} + \mathrm{i}\frac{\pi k}{\mathrm{Im}\,\tau} \mathrm{Im}\, z \right), \qquad a^\dagger = -\sqrt{\frac{\mathrm{Im}\,\tau}{\pi k}} \left(\frac{\partial}{\partial z} + \mathrm{i}\frac{\pi k}{\mathrm{Im}\,\tau} \mathrm{Im}\, z \right),$$

are the ∂- and $\bar{\partial}$-operators with respect to the complex structure defined by τ and k; and the Hamiltonian is the differential operator

$$H = \hbar\omega_c (a^\dagger a + \tfrac{1}{2}), \quad \omega_c = \frac{\hbar}{m_e} \frac{2\pi k}{\mathrm{Im}\,\tau}, \quad [a, a^+] = 1. \tag{3}$$

Notice that the a and $a^\dagger$ operators have been normalized in such a way that their commutator is 1 and m_e is the effective mass of the electron.

Eigensections of the Hamiltonian are the stationary states and the space $\Gamma(\Sigma, \mathcal{L}_k)$ decomposes as the direct sum of eigenspaces of H:

$$\Gamma(\Sigma, \mathcal{L}_k) = \bigoplus_N \Gamma_{E_N}(\Sigma, \mathcal{L}_k).$$

An element of $\Gamma_{E_N}(\Sigma, \mathcal{L}_k)$ can be understood as a function on $\mathbb{C}$ obeying the spectral equation

$$H\psi_N = E_N \psi_N \tag{4}$$

and the periodicity conditions:

$$\begin{aligned}&\psi_N(z+1,\bar z+1)=\psi_N(z,\bar z),\\&\psi_N(z+\tau,\bar z+\bar\tau)=\exp\{-2\pi ik\,\mathrm{Re}(z+\tfrac12\tau)\}\psi_N(z,\bar z).\end{aligned}\tag{5}$$

Mathematically, (5) guarantees that H acting on such a space of functions is a well-defined self-adjoint operator. On the physical side the second equation of (5) tells us that the magnetic field yields a Galileo transformation along the x_1-axis, projectively represented via a one-cocycle as usual, in the quantum space of states.

The first eigenvalue is $E_0=\frac12\hbar\omega_c$ and occurs for functions such that

$$\left(\frac{\partial}{\partial\bar z}+\mathrm{i}\frac{\pi k}{\mathrm{Im}\,\tau}\mathrm{Im}\,z\right)\psi_0(z,\bar z)=0.\tag{6}$$

One finds that the space of solutions of (6), usually called the first Landau level, complying with (5) is of dimension k. The set of functions

$$\psi_{0l}^{+}(z,\bar z)=\exp\left\{-\pi k\frac{(\mathrm{Im}\,z)^2}{\mathrm{Im}\,\tau}\right\}\Theta\begin{bmatrix}0\\ l/k\end{bmatrix}\left(z\,\middle|\,\frac{\tau}{k}\right),$$

$l=0,1,2,\dots,k-1$, where the Riemann Theta functions with characteristics are defined as, see [8],

$$\Theta\begin{bmatrix}a\\ b\end{bmatrix}(z\mid\tau)=\sum_{n\in\mathbb Z}\exp\left\{2\pi\mathrm{i}[(n+a)(z+b)+\frac12(n+a)^2\tau]\right\},$$

form a basis of $\Gamma_{E_0}(\Sigma,\mathcal L_k)$. In fact we can write ψ_{0l} in the form

$$\psi_{0l}(z,\bar z)=\exp\left\{-\pi k\frac{(\mathrm{Im}\,z)^2}{\mathrm{Im}\,\tau}\right\}f_l(z)$$

to see that the first Landau level is $H^0(\Sigma,\mathcal L_k)$, the space of holomorphic sections of $\mathcal L_k$:

$$f_l(z|\tau)=\Theta\begin{bmatrix}0\\ l/k\end{bmatrix}(z|\tau/k),\qquad l=0,1,\dots,k-1.$$

The higher Landau levels, the eigenspaces of eigenvalues $E_N=\hbar\omega_c(N+\frac12)$, $N>0$, are also of dimension k. A basis in $\Gamma_{E_N}(\Sigma,\mathcal L_k)$ is provided by

$$\begin{aligned}\psi_{Nl}(z,\bar z)=\mathrm e^{-(\pi k/\mathrm{Im}\,\tau)(\mathrm{Im}\,z)^2}&\sum_{n\in\mathbb Z}\mathrm e^{2\pi\mathrm in(z+l/k)}H_N\left(\sqrt{\frac{2\pi k}{\mathrm{Im}\,\tau}}\left(\mathrm{Im}\,z+\frac nk\right)\right)\\&\times\mathrm e^{-(\pi\,\mathrm{Im}\,\tau/k)n^2},\end{aligned}$$

where $H_N(y)$ is the Nth Hermitian polynomial.

2.2. The topological limit

The first 'Landau level' for the one-electron problem is the only surviving subspace of the Hilbert space when taking the topological limit, $m_e\to 0$. In the complex plane the topological limit amounts classically to a reduction of the phase space from $T^*\mathbb{C}$ to $\mathbb{C}$. Geometric quantization tells us how to deal with this process in the quantum domain.

Given a holomorphic function and two real numbers a and b, consider the transformations

$$S(b)f(z) = f(z+b), \qquad T(a)f(z) = \exp(\mathrm{i}\pi a^2 s + 2\pi\mathrm{i}az)f(z+as) \tag{7}$$

with the composition law

$$S(b_1)\circ S(b_2) = S(b_1+b_2), \qquad T(a_1)\circ T(a_2) = T(a_1+a_2).$$

The S and T operators do not commute with each other:

$$S(b)\circ T(a) = \mathrm{e}^{2\pi\mathrm{i}ab}T(a)\circ S(b),$$

i.e. there is a two-cocycle $\omega_2(z; a, b) = ab$. They form an honest unitary representation of the Heisenberg group $G_\mathrm{H} = \mathbb{C}_1^* \otimes \mathbb{R} \otimes \mathbb{R}$, $\mathbb{C}_1^* = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, with group law:

$$(\lambda, a, b)(\gamma, c, d) = (\lambda\gamma\exp(2\pi\mathrm{i}bc), a+c, b+d)$$

if $(\lambda, a, b) \in G_\mathrm{H}$ on the Hilbert space $\mathcal{H} = \{f \mid \|f\|^2 < +\infty\}$,

$$\|f\|^2 = \int_{\mathbb{C}} \mathrm{e}^{-2\pi x_2^2}|f(z)|^2\frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\mathrm{i}}$$

given by

$$U(\lambda, a, b)f(z) = \lambda(S(b)\circ T(a))f(z) = \lambda\mathrm{e}^{\pi\mathrm{i}a^2 s + 2\pi\mathrm{i}az}f(z+as+b).$$

Let us denote by Γ_1 the subgroup of G_H chosen as follows: $\Gamma_1 = \{(1, n_1, n_2) \in G_\mathrm{H}/ n_1, n_2 \in \mathbb{Z}\}$. Let $\hat{\Gamma}_{11}(1, n_2, n_2)$ be the corresponding irreducible unitary representation:

$$U(1, n_1, n_2)f(z) = T(n_1)\circ S(n_2)f(z) = \mathrm{e}^{\mathrm{i}\pi n_1^2 s + 2\pi\mathrm{i}n_1 z}f(z+n_1 s+n_2).$$

If we identify s with the modular parameter τ, which thus appears as a characteristic "complex" time of the system, the invariant subspace under $\hat{\Gamma}_{11}$ of $\mathcal{H}$,

$$\mathcal{H}^{11} = \{f(z) \in \mathcal{H}/U(1, n_1, n_2)f(z) = f(z) \text{ mod co-cycles}\},$$

is the first Landau level. $f(z) \in \mathcal{H}^{11}$ if and only if

$$f(z+1) = f(z), \qquad f(z+\tau) = \mathrm{e}^{-2\pi\mathrm{i}k(z+\tau/2)}f(z) \tag{8}$$

so that

$$f(z) = \sum_{l=0}^{k-1} c_l\Theta\begin{bmatrix}0\\ l/k\end{bmatrix}\left(z\,\middle|\,\frac{\tau}{k}\right)$$

and $\mathcal{H}^{11}$ is isomorphic to $H^0(\Sigma, \mathcal{L}_k)$.

For later use, another basis of $H^0(\Sigma, \mathcal{L}_k)$ is convenient: one can check that

$$\Theta\begin{bmatrix} 0 \\ l/k \end{bmatrix}\left(z|\frac{\tau}{k}\right) = \sum_{j=0}^{k-1} \exp\left(2\pi \mathrm{i}\frac{jl}{k}\right) \Theta\begin{bmatrix} j/k \\ 0 \end{bmatrix}(kz \mid k\tau)$$

and use the

$$g_l(z \mid \tau) = \Theta\begin{bmatrix} l/k) \\ 0 \end{bmatrix}(kz|k\tau)$$

as another basis in $H^0(\Sigma, \mathcal{L}_k)$:

$$(g_{l_1}, g_{l_2}) = \int \mathrm{e}^{-2\pi(\operatorname{Im} z)^2/\operatorname{Im}\tau} g^*_{l_1}(z) g_{l_2}(z) \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\mathrm{i}} = \delta_{l_1 l_2}.$$

More important: conditions (8) can be generalized to

$$f(z+1) = \mathrm{e}^{2\pi \mathrm{i}\phi_1} f(z), \qquad f(z+\tau) = \mathrm{e}^{2\pi \mathrm{i}\phi_2} \mathrm{e}^{-2\pi \mathrm{i}k(z+\tau/2)} f(z), \tag{9}$$

with $\phi_1, \phi_2 \in [0, 1]$. Physically it describes the double Aharanov–Bohm effect due to solenoid fluxes arranged as in the Thouless gedanken experiment, see Section 4. Mathematically, we deal with a family of line bundles $\mathcal{L}_k(\phi_1, \phi_2)$ parametrized by the Jacobian of Σ or the Picard group, the manifold of flat line bundles over Σ. A basis in $H^0(\Sigma, \mathcal{L}_k(\phi_1, \phi_2))$ is thus given by

$$f_l^{\phi}(z \mid \tau) = \Theta\begin{bmatrix} (l+\phi_1)/k \\ \phi_2 \end{bmatrix}(kz \mid k\tau), \quad l = 0, 1, \ldots, k-1,$$

which span the first Landau level. Physically, one can also understand the Jacobian as parametrizing "continuous" spin structures. This is not a problem in (2 + 1)-dimensions where both the spin and the statistics can be continuous.

In fact, the case $\phi_1 = \phi_2 = \frac{1}{2}$ is usually taken into account in the physics literature to describe odd wave functions. This choice would produce problems in the many-body wave functions because the use of addition formulas in Section 3 to compute the filling factor would require one to use simultaneously the $\phi = 0$ and $\phi = \frac{1}{2}$ cases, the analogs of the Ramond and Neveu–Schwartz fermionic sectors of string theory [13], which are disconnected. We thus prefer to stick to the more general perspective meant by $H^0(\Sigma, \mathcal{L}_k(\phi_1, \phi_2))$. To end this section, we comment on the fact that all the $\Gamma_{E_N}(\Sigma, \mathcal{L}_k)$ eigenspaces are isomorphic; they are thus probably formed by holomorphic sections of a bundle, also of degree k, with a different complex structure.

3. The ground state of the fractional quantum Hall effect

3.1. Laughlin state on a torus: Haldane–Rezayi wave functions

The nature of the ground state of a two-dimensional gas of electrons lies at the heart of the quantum Hall effect. The low temperatures and energies involved suggest that only the first Landau level enters the many-body ground state wave function. For a Hall device topologically equivalent to an open disc Laughlin [4] proposed the following wave function:

$$\Psi_{\mathrm{L}} \equiv \prod_{i<j}(z_i - z_j)^m \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} \mid z_i \mid^2\right\}. \tag{10}$$

We take z_i as the non-dimensional coordinate for the ith electron; $m = 2p + 1$ is an odd positive integer, and N the total number of electrons in the sample. Ψ_{L} is a precise combination of Slater determinants, the Vandermonde determinant if $m = 1$, and the number of one-electron occupied states is $N_{\mathrm{L}} = m(N-1)+1$. Notice that the first Landau level, in the symmetric gauge, is formed by linear combinations of the monomials z^q if we consider the problem in the complex plane $\mathbb{C}$. N_L is read from repeated use of the binomial formula and the filling factor $\nu_{\mathrm{L}} = N/N_{\mathrm{L}}$ is $1/m = 1/(2p+1)$ when $N \to \infty$. If $\sigma_{\mathrm{H}} = (\mathrm{e}^2/h)\nu_{\mathrm{L}}$, this provides an explanation for the 'rational' quantification of $(h/\mathrm{e}^2)\sigma_{\mathrm{H}}$ in the fractional quantum Hall effect. There is an overwhelming evidence that Ψ_{L} is a very good variational ground state for the many-body Hall Hamiltonian when two-body electron interactions are included. Moreover, a whole hierarchy of filling factors and plateaux centers are obtained by also considering quasi-particle and quasi-hole wave functions:

$$\begin{aligned} &\Psi_{\mathrm{qe}} \equiv \prod_i (2\partial_z - \bar{z}_i)\Psi_{\mathrm{L}}\mathrm{e}^{-|z|^2/2}, \qquad \Psi_{\mathrm{qh}} \equiv \prod_i (z - z_i)\Psi_{\mathrm{L}}\mathrm{e}^{-|z|^2/2}, \\ &\nu_{\mathrm{qe}} = \frac{n_{\mathrm{qe}}}{2pn_{\mathrm{qe}} \pm 1}, \qquad \nu_{\mathrm{qh}} = 1 - \frac{n_{\mathrm{qh}}}{2pn_{\mathrm{qh}} \pm 1}. \end{aligned} \tag{11}$$

There is one drawback however: Ψ_{L} provides a non-degenerate ground state and there is no generalized Thouless formula [5], see Section 4, which shows σ_{H} in terms of the ground state of the system as a topological invariant, independent of the perturbations induced by adiabatically changing solenoid fluxes: there is no reason to keep fixed σ_{H} when ν_{H} changes and no explanation for the plateaux is provided merely by identifying the ground state of the FQHE.

In the case where the two-dimensional many-electron Hall fluid moves in a periodic lattice, Haldane and Rezayi[10] have shown how to generalize Laughlin's variational state. The many-body Hamiltonian is

$$H = \frac{1}{m_e}\sum_{i=1}^{N}\left(a_i^{\dagger}a_i + \frac{\pi k}{\operatorname{Im}\tau}\right), \tag{12}$$

with a_i the differential operator of Section 2.1 for the ith electron. Denote by $m = 2p+1 = k/N$, a positive odd integer number, the ratio of k to the number of particles N. According to previous experience the candidates to become the ground state should be analytical functions in the z_i's, apart from the prefactor that may be included in the integration measure of the L^2-hermitian metric, satisfying the periodicity conditions:

$$\begin{aligned} \Psi(z_1, \ldots, z_i + 1, \ldots, z_N) &= (-1)^{N-1}\mathrm{e}^{2\pi\mathrm{i}\phi_1}\Psi(z_1, \ldots, z_i, \ldots, z_N), \\ \Psi(z_1, \ldots, z_i + \tau, \ldots, z_N) &= (-1)^{N-1}\mathrm{e}^{-2\pi\mathrm{i}\phi_2}\exp\{-2\pi\mathrm{i}k\,\mathrm{Re}(z_i + \tfrac{1}{2}\tau)\} \\ &\quad \times\Psi(z_1, \ldots, z_i, \ldots, z_N) \end{aligned} \tag{13}$$

for every i. We allow for arbitrary phases which mathematically amounts to tensoring with flat bundles. By doing this, we take into account the physical effect of the solenoid fluxes encircled by the homology-generating one-cycles.

Translational invariance in the complex plane case calls for the use of center of mass, $Z = \sum_{i=1}^{N} z_i$, and relative, $z_{ij} = z_i - z_j$, coordinates, Haldane and Rezayi, mimicking the Laughlin–Jastrow ansatz of the non-periodic case, proposed the following many-body wave function as the ground state of the FQHE on a periodic lattice:

$$\begin{aligned}\Psi(z_1, z_2, \ldots, z_N) &= F_{\mathrm{HR}}(z_1, z_2, \ldots, z_N) \mathrm{e}^{-\pi k/\mathrm{Im}\,\tau \sum_{i=1}^{N} (\mathrm{Im}\, z_i)^2}, \\ F_{\mathrm{HR}}(z_1, z_2, \ldots, z_N) &= F(Z) \prod_{i<j} f(z_{ij}).\end{aligned} \tag{14}$$

In order to satisfy Fermi statistics and comply with conditions (13), the relative wave functions must be odd when changing z_{ij} to $-z_{ij}$ and must satisfy

$$f(z_{ij} + 1) = \eta_1 f(z_{ij}), \qquad f(z_{ij} + \tau) = \eta_2 \exp(2\pi \mathrm{i} m z_{ij}) f(z_{ij}), \tag{15}$$

where η_1 and η_2 are constant phases. The choice of Haldane–Rezayi is

$$f(z_{ij}) = \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_{ij}|\tau) \tag{16}$$

a section in $H^0(\Sigma(i,j), \mathcal{L}_1^{\otimes m})$: $c_1(\mathcal{L}_1^{\otimes m}) = (1/2\pi\mathrm{i}) \int_{\Sigma} \mathrm{d}\log f = m$. From (13)–(16) one obtains

$$\begin{aligned}F(Z+1) &= \mathrm{e}^{\mathrm{i}2\pi\phi_1} F(Z), \\ F(Z+\tau) &= \mathrm{e}^{-\mathrm{i}2\pi\phi_2} \exp(2\pi \mathrm{i} m (Z + \tfrac{1}{2}\tau)) F(Z).\end{aligned} \tag{17}$$

Thus

$$F_l(Z) = \Theta \begin{bmatrix} (l+\phi_1)/m \\ \phi_2 \end{bmatrix} (mZ|m\tau) \tag{18}$$

solves (17). The Haldane–Rezayi wave function

$$F_l^{\mathrm{HR}} \equiv \Theta \begin{bmatrix} (l+\phi_1)/m \\ \phi_2 \end{bmatrix} (mZ|m\tau) \prod_{i<j} \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_{ij}|\tau) \tag{19}$$

is m times degenerated. The degeneracy is due to the center of mass dynamics which is not trivial when the configuration space is compact.

A very important ingredient of the variational approach to the FQHE in the complex plane is lacking in the generalization to a periodic lattice: there is no analog to the expansion of the Laughlin wave function as a linear combination of Slater determinants, explicitly used in the physics literature. Therefore, it is not possible to make a direct computation of the filling factor corresponding to the Haldane–Rezayi ground state. Related to this problem are difficulties in the definition of quasi-particle and quasi-hole states on a periodical lattice in a logical way.

We shall develop the Haldane–Rezayi wave function in terms of one-particle Landau states in the next section: the use of addition formulas will lead us to obtain the filling factor in a precise way. By extending this procedure we shall deal with the definition of quasi-particle wave functions in a periodic lattice, still an open question in the theory. A fine point that we shall address in our derivation is the problem of passing to CM and relative coördinates. On a genus 1 Riemann surface this is not an injective change of coordinates, but a $2N$ to 1 map, an isogeny between Cartesian products of elliptic curves.

3.2. Addition formulas

In this section we shall offer a geometric formulation of the Haldane–Rezayi wave function (19) for the ground state of the fractional quantum Hall effect in the periodic case. There are two main results: firstly the proof that the Haldane–Rezayi wave functions are unique verifying certain natural invariance conditions; secondly, the expression of the Haldane–Rezayi wave function in terms of one-particle states leading to an explicit computation of the filling factors in the periodic case. Both points are skipped in the physics literature about the FQHE on a periodic lattice but focus on the problems mentioned in the previous section. In particular, the use of CM and relative coordinates for a system of N particles moving in $\mathbb{C}$ is convenient when the system enjoys translational symmetry. In a periodic lattice there is only symmetry under a subgroup of the translation group: one would expect difficulties in using an ansatz such as (14) in the periodic case. We shall develop with precision the cases of $N = 2, 3$ electrons. In the general case, we shall give the explicit formulas. The proof will appear elsewhere [14].

We recall from Section 2.1 that the first Landau level in the periodic case is given by the vector space $H^0(\Sigma, \mathcal{L}_k)$. Assuming that an origin $e \in \Sigma$ for the group law of Σ is fixed and that $\mathcal{L}_k = \mathcal{O}_\Sigma(ke)$, then the holomorphic global sections $H^0(\Sigma, \mathcal{L}_k)$ are identified with the vector space of kth-order Theta functions of the elliptic curve Σ which are the one-particle states in the topological limit.

3.2.1. $N = 2$ electrons

The phase space of a system of two electrons in the topological limit is the product surface $\Sigma \times \Sigma$. The line bundle accounting for the fact that they are in the presence of a magnetic field compatible with the lattice is

$$M_2 = p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k$$

$p_i : \Sigma \times \Sigma \to \Sigma$ being the two natural projections.

Let $V_k = H^0(\Sigma, \mathcal{L}_k)$ be the vector space of kth-order Theta fuctions. By the Künneth formula we have

$$H^0(\Sigma \times \Sigma, M_2) = V_k \otimes_{\mathbb{C}} V_k.$$

The quantum space of states of two electrons moving in a periodic lattice in the topological limit is a certain subspace of $V_k \otimes V_k$ which we are going to characterize. The

Laughlin–Jastrow variational approach to the FQHE of $N = 2$ electrons in the complex plane is based on the change of variables given by the equations

$$(z_1, z_2) \longrightarrow (z_1 + z_2, z_1 - z_2)$$

The Haldane–Rezayi generalization to the periodic system makes use of a similar mapping. The main difference is that the above equations do not define a change of variables, a one-to-one map, in $\Sigma \times \Sigma$. With precision, let us consider the morphism of surfaces:

$$\xi : \Sigma \times \Sigma \longrightarrow \Sigma \times \Sigma$$

defined by $\xi(z_1, z_2) = (z_1 + z_2, z_1 - z_2)$. It is well known that ξ is a finite morphism of degree 4; i.e. ξ is an isogeny of abelian varieties and the kernel of ξ is

$$\Sigma_2 = \{z \in \Sigma \text{ such that } 2z = 0 \text{ in } \Sigma\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

immersed in $\Sigma \times \Sigma$ by the diagonal.

A fundamental element of our formulation is the geometric version of the classical addition formula which describes the pull-back of a line bundle over $\Sigma \times \Sigma$ by the isogeny ξ:

Addition Formula (Mumford [15]). *Let $\mathcal{L}$ be a symmetric line bundle over Σ,* $((-1)^*\mathcal{L} = \mathcal{L})$. *There exists an isomorphism of line bundles over $\Sigma \times \Sigma$:*

$$\xi^*(p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) \cong (p_1^*\mathcal{L} \otimes p_2^*\mathcal{L})^{\otimes 2}.$$

As a consequence, we deduce the existence of a homomorphism between the vector spaces of holomorphic sections:

$$\xi^* : H^0(\Sigma, \mathcal{L}) \otimes H^0(\Sigma, \mathcal{L}) \longrightarrow H^0(\Sigma, \mathcal{L}^{\otimes 2}) \otimes H^0(\Sigma, \mathcal{L}^{\otimes 2})$$

such that given $\theta, \eta \in H^0(\Sigma, \mathcal{L})$:

$$\xi^*(\theta(z_1) \otimes \eta(z_2)) = \theta(z_1 + z_2)\eta(z_1 - z_2)$$

and given a basis $\{\theta_i\}$ of $H^0(\Sigma, \mathcal{L}^{\otimes 2})$ we obtain the relation

$$\theta(z_1 + z_2)\eta(z_1 - z_2) = \sum \lambda_{ij}\theta_i(z_1)\theta_j(z_2). \tag{20}$$

In the FQHE on a periodic lattice the natural invariance conditions to impose on the wave functions are restricted to the subgroup

$$\Sigma_2 \overset{\Delta}{\hookrightarrow} \Sigma \times \Sigma,$$

$\Delta(\alpha) = (\alpha, \alpha)$ is the embedding by the diagonal of $\alpha \in \Sigma_2$. This in turn requires that $p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k$ should be invariant by $\Delta(\Sigma_2)$, which happens if and only if $k = 2m$ for a positive integer m.

Application of the addition formula and the above considerations to the line bundle $\mathcal{L}_m = \mathcal{O}_\Sigma(me)$ lead to the following results:

1. There exists an isomorphism of line bundles

$$\xi^*(p_1^*\mathcal{L}_m \otimes p_2^*\mathcal{L}_m) \cong p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k,$$

with $k = 2m$.

2. The line bundle $p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k$ is invariant by $\Delta(\Sigma_2)$. Moreover, under these invariance conditions, there is a homomorphism of vector spaces:

$$\xi^* : V_m \otimes V_m \to V_k \otimes V_k,$$

where $V_m = H^0(\Sigma, \mathcal{L}_m)$, $V_k = H^0(\Sigma, \mathcal{L}_k)$.

Let $E_k(2)$ be the quantum vector space of the two-electron system occuping states in the first Landau level. We assume that the vectors in $E_k(2)$ satisfy, on one hand, the translation invariance condition with respect to the two-torsion, Σ_2, and on the other hand, fermionic statistics. The first assumption tells us that $E_k(2)$ is a vector subspace of the image of the homomorphism ξ^*, $\mathrm{Im}\,\xi^*$. The second condition about the statistics means that $E_k(2)$ is a subspace of $\Lambda^2 V_k$, the second exterior power of V_k:

$$E_k(2) = \mathrm{Im}(\xi^*) \cap \Lambda^2 V_k \subset V_k \otimes V_k.$$

Given our intrinsic characterization of the vector space of wave functions as a subspace of $V_k \otimes V_k$, we can compute it explicitly:

Let $V_m^{\pm}$ be the subspaces of V_m formed by eigenvectors of the action of the automorphism $[-1] : \Sigma \to \Sigma$, $[-1](z) = -z$. The vector space of wave functions is $E_k(2) = \xi^*(V_m \otimes V_m^-)$ and a basis in $E_k(2)$ is given by

$$\theta_i(z_1 + z_2)\theta_{j_-}^-(z_1 - z_2); \quad i = 1, 2, \ldots, m, \quad j_- = 1, 2, \ldots, \tfrac{1}{2}(m-1),$$

where

$$\left\{\theta_i(z) = \Theta\begin{bmatrix} i/m \\ 0 \end{bmatrix}(mz \mid m\tau)\right\}$$

is a basis of V_m and

$$\left\{\theta_{j_-}^-(z) = \frac{1}{2}\left(\Theta\begin{bmatrix} j_-/m \\ 0 \end{bmatrix}(mz|m\tau) - \Theta\begin{bmatrix} -j_-/m \\ 0 \end{bmatrix}(mz|m\tau)\right)\right\}$$

is a basis of V_m^-.

Proposition 1. *For each wave fuction $\theta_i(z_1 + z_2)\theta_{j_-}^-(z_1 - z_2) \in E_k(2)$ it is verified that*

$$\theta_i(z_1 + z_2)\theta_{j_-}^-(z_1 - z_2) = \sum_{i<j} \lambda_{ij}(\theta_i^{(k)}(z_1)\theta_j^{(k)}(z_2) - \theta_j^{(k)}(z_1)\theta_i^{(k)}(z_2)), \tag{21}$$

where

$$\left\{\theta_i^{(k)}(z) = \Theta\begin{bmatrix} i/k \\ 0 \end{bmatrix}(kz \mid k\tau), \quad i = 1, 2, \ldots, k\right\}$$

is a basis of V_k.

This is an inmediate consequence of the identity $E_k(2) = \xi^*(V_m \otimes V_m^-)$. Note that this result is the analog in the periodic case of the development in terms of Slater determinants of the N-particle wave functions used in the complex plane case.

The Haldane–Rezayi wave function (19), for $\phi_1 = \phi_2 = 0$,

$$F_l^{\mathrm{HR}}(z_1, z_2) = \theta_l(z_1 + z_2) \sum_{j_-=1}^{p=(m-1)/2} c_{j_-}\theta_{j_-}^-(z_1 - z_2),$$

$l = 1, 2, \ldots, m$, includes a special linear combination

$$\theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z_1 - z_2|\tau) = \sum_{j_-} c_{j_-}\theta_{j_-}^-(z_1 - z_2),$$

with a single zero of order m at $z_1 - z_2 = 0$ and no other zeros away from the diagonal.

The important physical point is that Proposition 1 allows us to write a general wave function $F(z_1, z_2)$ in the form

$$F(z_1, z_2) = \sum_{i<j} \lambda_{ij}(\theta_i^{(k)}(z_1)\theta_j^{(k)}(z_2) - \theta_j^{(k)}(z_1)\theta_i^{(k)}(z_2)). \tag{22}$$

In particular, this expression is valid for the Haldane–Rezayi wave function $F_l^{\mathrm{HR}}(z_1, z_2)$ for a specific set of constants. Using (22), the Haldane–Rezayi wave function can be given explicitly in terms of the one-particle quantum states belonging to the first Landau level. There are k states to be occupied by two particles in one anti-symmetric combination. The filling factor of the Haldane–Rezayi ground state is thus: $\nu_{\mathrm{HR}} = 2/k = 1/m$. This is an obvious consequence of Proposition 1 since $\dim V_k$ is $k = 2m$.

3.2.2. $N = 3$ electrons

The system is now described by the line bundle

$$M_3 = p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k \otimes p_3^*\mathcal{L}_k$$

over $\Sigma \times \Sigma \times \Sigma$, $p_i : \Sigma \times \Sigma \times \Sigma \to \Sigma$ being the natural projections. Let $\Sigma_3 = \{z \in \Sigma$ such that $3z = 0\}$ be the subgroup of three-torsion points of Σ and let us consider the diagonal immersion

$$\begin{aligned} &\Sigma_3 \xrightarrow{\Delta} \Sigma \times \Sigma \times \Sigma \\ &\alpha \longrightarrow (\alpha, \alpha, \alpha). \end{aligned}$$

We impose on the system the condition of translation invariance with respect to the group $\Delta(\Sigma_3)$. M_3 is invariant by $\Delta(\Sigma_3)$ if and only if $k = 3m$ for a positive integer m and this condition is also equivalent to the existence of an isomorphism $\mathcal{L}_k \simeq \mathcal{L}_m^{\otimes 3}$. For physical reasons we shall assume that $m = 2p \pm 1$ ($p \leq 1$, $p \in \mathbb{Z}^+$).

A parallel analysis to the $N = 2$ case requires the formula of the Cube (Mumford [8]):

Let $m : \Sigma \times \Sigma \times \Sigma \to \Sigma$ be the addition map, $m_{ij} : \Sigma \times \Sigma \times \Sigma \to \Sigma$, the partial addition maps and $p_i : \Sigma \times \Sigma \times \Sigma \to \Sigma$ the three projections:

$$m(z_1, z_2, z_3) = z_1 + z_2 + z_3,$$
$$m_{ij}(z_1, z_2, z_3) = z_i + z_j, \qquad p_i(z_1, z_2, z_3) = z_i.$$

Given a line bundle $\mathcal{L}$ over Σ, there exists a natural isomorphism

$$m^*\mathcal{L} \cong m_{12}^*\mathcal{L} \otimes m_{13}^*\mathcal{L} \otimes m_{23}^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \otimes p_3^*\mathcal{L}^{-1} \tag{23}$$

between line bundles over $\Sigma \times \Sigma \times \Sigma$.

Moreover, denoting by $s_{ij} : \Sigma \times \Sigma \times \Sigma \to \Sigma$ the subtraction morphisms, $s_{ij}(z_1, z_2, z_3) = z_i - z_j$, there exists an isomorphism between line bundles on $\Sigma \times \Sigma \times \Sigma$:

$$m^*\mathcal{L} \otimes s_{12}^*\mathcal{L} \otimes s_{13}^*\mathcal{L} \otimes s_{23}^*\mathcal{L} \cong p_1^*\mathcal{L}^{\otimes 3} \otimes p_2^*\mathcal{L}^{\otimes 3} \otimes p_3^*\mathcal{L}^{\otimes 3}. \tag{24}$$

This follows from the isomorphism $m_{ij}^*\mathcal{L} \otimes s_{ij}^*\mathcal{L} \cong p_i^*\mathcal{L}^{\otimes 2} \otimes p_j^*\mathcal{L}^{\otimes 2}$ and the Cube formula (23). Applying the above isomorphism (24) to the line bundle, $M_3 = p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k \otimes p_3^*\mathcal{L}_k$ invariant with respect $\Delta(\Sigma_3)$

$$\begin{aligned} &H^0(\Sigma, \mathcal{L}_k) \otimes H^0(\Sigma, \mathcal{L}_k) \otimes H^0(\Sigma, \mathcal{L}_k) \\ &\quad \cong H^0(\Sigma \times \Sigma \times \Sigma, m^*\mathcal{L}_m \otimes s_{12}^*\mathcal{L}_m \otimes s_{13}^*\mathcal{L}_m \otimes s_{23}^*\mathcal{L}_m). \end{aligned}$$

Fermionic statistics means that the vector space $E_k(3)$ of wave functions is contained in the vector subspace $\Lambda^3 V_k$ of $V_k \otimes V_k \otimes V_k$. To deal with $\Delta(\Sigma_3)$ translation invariance, consider the morphism

$$\begin{aligned} &\xi_3 : \Sigma \times \Sigma \times \Sigma \longrightarrow \Sigma \times \Sigma \times \Sigma \times \Sigma \\ &\xi_3(z_1, z_2, z_3) = (z_1 + z_2 + z_3, z_1 - z_2, z_1 - z_3, z_2 - z_3). \end{aligned}$$

The kernel of ξ_3 is $\Delta(\Sigma_3) \subset \Sigma \times \Sigma \times \Sigma$ and it is verified that

$$\xi_3^*(p_1^*\mathcal{L}_m \otimes p_2^*\mathcal{L}_m \otimes p_3^*\mathcal{L}_m \otimes p_4^*\mathcal{L}_m) \cong m^*\mathcal{L}_m \otimes s_{12}^*\mathcal{L}_m \otimes s_{13}^*\mathcal{L}_m \otimes s_{23}^*\mathcal{L}_m.$$

There is a homomorphism between global sections:

$$\begin{aligned} &H^0\left(\Sigma \times \Sigma \times \Sigma \times \Sigma, \bigotimes_{i=1}^{4} p_i^*\mathcal{L}_m\right) \\ &\quad = V_m \otimes V_m \otimes V_m \otimes V_m \xrightarrow{\xi_3^*} H^0(\Sigma \times \Sigma \times \Sigma \times \Sigma, M_3) = V_k \otimes V_k \otimes V_k. \end{aligned}$$

Invariance under $\Delta(\Sigma_3)$ means that $E_k(3) \subset \mathrm{Im}(\xi_3^*)$ and

$$E_k(3) = \mathrm{Im}(\xi_3^*) \cap \Lambda^3 V_k \subset V_k \otimes V_k \otimes V_k$$

is the quantum space of states of the three-particle system. Also, using the homomorphism between global sections above, it can be proved that

$$E_k(3) = \xi_3^*(V_m \otimes V_m^- \otimes V_m^- \otimes V_m^-).$$

ξ_3 is a morphism between varieties of different dimension, and hence it is very difficult to compute its equations explicitly. The solution to this problem will be to show that ξ_3 factors through an isogeny and then to apply the well-known results [15] about the explicit equations of the homomorphisms induced by isogenies between the vector spaces of global sections.

The factorization of ξ_3 through an isogeny is given by the commutative diagram

$$\begin{array}{ccc} \Sigma\times\Sigma\times\Sigma & \xrightarrow{\xi_3} & \Sigma\times\Sigma\times\Sigma\times\Sigma \\ & \varphi_3 \searrow & \downarrow \Pi_{123} \\ & & \Sigma\times\Sigma\times\Sigma \end{array}$$

where $\varphi_3(z_1, z_2, z_3) = (z_1 + z_2 + z_3, z_1 - z_2, z_1 - z_3)$ and Π_{123} is the projection into the first three factors. Let us define a line bundle N_3 on $\Sigma\times\Sigma\times\Sigma$ by

$$N_3 = p_1^*\mathcal{L}_m \otimes p_2^*\mathcal{L}_m \otimes p_3^*\mathcal{L}_m \otimes s_{32}^*\mathcal{L}_m.$$

One has that $\xi_3^*(\otimes_{i=1}^4 p_i^*\mathcal{L}_m) = \varphi_3^* N_3 = M_3$. Then, $\mathrm{Im}(\xi_3^*) = \mathrm{Im}(\varphi_3^*)$, and the computation of ξ_3^* is reduced to computing φ_3^*. But φ_3 is an isogeny and by applying the results on the behavior of global sections under isogenies [15] we finally obtain

$$\begin{aligned} &\varphi_3^*\left(\theta[d_1](z_1) \otimes \theta[d_2](z_2) \otimes \theta[d_3](z_3) \otimes s_{32}^*(\theta[d_3 - d_2](z))\right) \\ &\quad = \theta[d_1](z_1 + z_2 + z_3)\theta[d_2](z_1 - z_2)\theta[d_3](z_1 - z_3)\theta[d_3 - d_2](z_2 - z_3) \\ &\quad = \lambda \sum_{b_i \in \mathbb{Z}_k} \theta[b_1](z_1)\theta[b_2](z_2)\theta[b_3](z_3) \end{aligned} \tag{25}$$

Here

$$\theta[d](z) = \Theta\begin{bmatrix} d/3m \\ 0 \end{bmatrix}(mz|m\tau), \qquad d = 3, 6, 9, \ldots, 3m = k,$$

$$\theta[b](z) = \Theta\begin{bmatrix} b/k \\ 0 \end{bmatrix}(kz \mid k\tau), \quad b = 1, 2, \ldots, k$$

and $b_1 + b_2 + b_3 \equiv d_1 \bmod k$, $b_1 - b_2 \equiv d_2 \bmod k$, $b_1 - b_3 \equiv d_3 \bmod k$. $\lambda \in \mathbb{C}$ is a constant independent of (d_1, d_2, d_3).

A basis for the wave functions in $E_k(3)$ has the form

$$\begin{aligned} &\theta[d_1](z_1 + z_2 + z_3)\theta_-[d_2^-](z_1 - z_2)\theta_-[d_3^-](z_1 - z_3) \\ &\qquad \times\ \theta_-[d_3^- - d_2^-](z_2 - z_3) \in E_k(3), \end{aligned}$$

where $\theta_-[d^-](z) = \frac{1}{2}(\theta[d^-](z) - \theta[-d^-](z))$, $d^- = 1, 2, \ldots, \frac{1}{2}(m-1) = p$. The Haldane–Rezayi wave function, m times degenerated, is

$$F^{\mathrm{HR}}_{d_1/3}(z_1, z_2, z_3) = \theta[d_1](z_1 + z_2 + z_3)\theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_1 - z_2|\tau)\theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \times (z_1 - z_3|\tau)\theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_2 - z_3|\tau),$$

which can be expresed as a state in $E_k(3)$ by using

$$\theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_i - z_j) = \sum_{c_{d^-}=1}^{p} c_{d^-}\theta_-[d^-](z_i - z_j).$$

$F^{\mathrm{HR}}_{d_1/3}$ are special states characterized by having zeros of order m when $z_i - z_j = 0$ and no other zeros away from the diagonal. Using (25) the expansion

$$F^{\mathrm{HR}}_{d_1/3}(z_1, z_2, z_3) = \sum_{i_1<i_2<i_3} \lambda_{i_1 i_2 i_3} H(\theta[b_{i_1}](z_1)\theta[b_{i_2}](z_2)\theta[b_{i_3}](z_3)) \tag{26}$$

decomposes the Haldane–Rezayi wave function into one-particle states. H is the antisymetization operator

$$H(\theta[b_{i_1}](z_1)\theta[b_{i_2}](z_2)\theta[b_{i_3}](z_3)) = \sum_{\sigma\in S_3} \mathrm{sg}(\sigma)\theta[b_{\sigma(i_1)}](z_1)\theta[b_{\sigma(i_2)}](z_2)\theta[b_{\sigma(i_3)}](z_3)$$

and $\lambda_{i_1 i_2 i_3} \in \mathbb{C}$. The filling factor is again $\nu_{\mathrm{HR}} = 3/k = 1/m$.

3.2.3. N electrons

The system is defined by the line bundle

$$M_N = p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k \otimes \cdots \otimes p_N^*\mathcal{L}_k$$

over $\Sigma \times \Sigma \times \cdots \times \Sigma (N \text{times})$. The translation invariance condition under $\Delta(\Sigma_N)$ (Σ_N being the N-torsion subgroup of Σ) is equivalent to saying that $k = Nm$. Let us consider the morphism $\varphi_N : \Sigma^N \to \Sigma^N$ given by

$$\varphi_N(z_1, z_2, \ldots, z_N) = (z_1 + z_2 + \cdots + z_N, z_1 - z_2, \ldots, z_1 - z_N),$$

φ_N is a finite morphism of kernel equal to $\Sigma_N \xrightarrow{\Delta} \Sigma^N$. Let N_N be the line bundle over Σ^N defined by

$$N_N = p_1^*\mathcal{L}_m \otimes \cdots \otimes p_N^*\mathcal{L}_m \otimes \left(\bigotimes_{\substack{i>j \\ j\geq 2}} s_{ij}^*\mathcal{L}_m \right).$$

One proves that $\varphi_N^* N_N \cong M_N = \otimes_{i=1}^N p_i^*\mathcal{L}_k$ for $k = mN$. There is a homomorphism between global sections:

$$\varphi_N^* : H^0(\Sigma^N, N_N) \longrightarrow \underbrace{V_k \otimes \cdots \otimes V_k}_{N},$$

and arguing as in the $N = 2, 3$ cases, we deduce that the vector space $E_k(N)$ of wave functions is

$$E_k(N) = \mathrm{Im}(\varphi_N^*) \cap \Lambda^N V_k \subset V_k \underbrace{\otimes \cdots \otimes}\, V_k,$$

and moreover,

$$E_k(N) = \varphi_N^*(V_m \otimes V_m^- \underbrace{\otimes \cdots \otimes}_{N-1} V_m^-).$$

The fermionic Fock space of the system is thus

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} E_k(N).$$

Again the Haldane–Rezayi ground state admits an expansion in terms of one-particle wave functions living in V_k,

$$\begin{aligned} &\theta[d](z_1 + z_2 + \cdots + z_N) \prod_{i<j} \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_i - z_j | \tau) \\ &\quad = \sum_{i_1 < \cdots < i_N} \lambda_{i_1 \cdots i_N} H(\theta[b_{i_1}](z_1) \cdots \theta[b_{i_N}] z_N) \end{aligned} \tag{27}$$

where

$$\theta[d] \in V_m, \qquad \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \in V_m^-, \qquad \theta[b] \in V_k,$$

H is the antisymmetrization operator and $\lambda_{i_1 \cdots i_N}$ are constants. The HR filling factor is thus $\nu_{\mathrm{HR}} = N/k = 1/m$. Note that the general filling factor for a generic vector state in $E_k(N)$ is also $\nu_{\mathrm{G}} = N/k$.

3.3. *Quasi-holes and quasi-particles*

Quasi-hole and quasi-particle excitations to the ground state of the FQHE in the complex plane were identified as having an important role by Laughlin [4]. Haldane and Rezayi [10] also define quasi-hole and quasi-particle wave functions in a periodic lattice. In this section we shall develop a geometric formulation of the states mentioned above with the aim of clarifying some obscure points and difficulties arising in this matter. The main difference in our approach is that we do not define quasi-holes and quasi-particles by the action of appropriate operators on the Haldane–Rezayi ground state; instead, we adopt a broader point of view and focus on defining the spaces of wave functions of many-electron system including quasi-holes and quasi-particles intrinsically.

3.3.1. *Geometric formulation of a system with quasi-holes*

We study three significant cases:

3.3.1.1. Two electrons and one quasi-hole. Let us consider the set of morphisms:

$$\begin{array}{ccc} \Sigma\times\Sigma\times\Sigma & \xrightarrow{\xi_{12}} & \Sigma\times\Sigma\times\Sigma \\ s_{10}\swarrow \quad \searrow s_{20} & & \\ \Sigma \qquad \Sigma & & \end{array}$$

defined by:

$$s_{10}(z_0,z_1,z_2)=z_1-z_0,\, s_{20}(z_0,z_1,z_2)=z_2-z_0$$

and

$$\xi_{12}(z_0,z_1,z_2)=(z_0,z_1+z_2,z_1-z_2).$$

The system of two electrons and one quasi-hole is described by the line bundle

$$M_{2,1}=p_1^*\mathcal{L}_k\otimes p_2^*\mathcal{L}_k\otimes s_{10}^*\mathcal{L}_1\otimes s_{20}^*\mathcal{L}_1$$

as follows: if $k=2m$, there is a translation-invariance with respect to the two-torsion $\Delta_{12}(\Sigma_2)$, $\Delta_{12}(\alpha)=(0,\alpha,\alpha)$, $\alpha\in\Delta_{12}(\Sigma_2)$, and hence

$$M_{2,1}\cong\xi_{12}^*(p_1^*\mathcal{L}_m\otimes p_2^*\mathcal{L}_m)\otimes s_{10}^*\mathcal{L}_1\otimes s_{20}^*\mathcal{L}_1.$$

The change of variables in $\Sigma\times\Sigma\times\Sigma$ defined by $\sigma(z_0,z_1,z_2)=(-z_0,z_1,z_2)$ makes explicit computations easier; we obtain the line bundle

$$N_{2,1}\equiv\sigma^*M_{2,1}=p_1^*\mathcal{L}_m^{\otimes 2}\otimes p_2^*\mathcal{L}_m^{\otimes 2}\otimes m_{10}^*\mathcal{L}_1\otimes m_{20}^*\mathcal{L}_1,$$

where $m_{10}(z_0,z_1,z_2)=z_0+z_1$, $m_{20}(z_0,z_1z_2)=z_0+z_2$. The quantum space of states of two electrons and a hole is a subspace $E_k(2,1)$ of the space of holomorphic sections of $N_{2,1}$. $H^0(\Sigma\times\Sigma\times\Sigma, N_{2,1})$. The wave functions for the quasi-hole are the sections in $E_k(2,1)$. In particular, the Haldane–Rezayi wave function is

$$\begin{aligned} F_l^{\mathrm{QH}}(z_0,z_1,z_2)=&\,\Theta\begin{bmatrix} l/m \\ 0\end{bmatrix}(z_0+m(z_1+z_2)|m\tau)\Theta^m\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_1-z_2|\tau) \\ &\times\Theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_1-z_0|\tau)\Theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_2-z_0|\tau). \end{aligned} \tag{28}$$

The quasi-hole creation operator at the point $z_0\in\Sigma$ corresponds to multiplication by

$$\prod_{i=1,2}\Theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_i-z_0|\tau)$$

of the ground state wave function. Observe that there is also a correction in the center of mass contribution; the QH coordinate z_0 enters the CM wave function with a different scale factor as compared to the particle coordinates. This produces differences in the holonomy with respect to particle holonomies in the ground state leading to fractionary charge and statistics as will be discussed in the sequel.

There is another physically interesting point of view, see [3]. Integration on the electron coordinates, z_1,z_2 in (28), yields an "effective" wave function for the quasi-hole showing their physically relevant features. Geometrically, this amounts to taking the direct image with respect to $p_0 : \Sigma \times \Sigma \times \Sigma \to \Sigma$, $(p_0(z_0, z_1, z_2) = z_0)$, and considering the bundle $W_{2,1} = p_{0*}N_{2,1}$. Effective wave functions live in a subspace of $H^0(\Sigma, W_{2,1})$ but the interesting identity

$$H^0(\Sigma \times \Sigma \times \Sigma, N_{2,1}) = H^0(\Sigma, W_{2,1}) \tag{29}$$

holds. A subtle point is the following: $N_{2,1}$ is a line bundle but $W_{2,1}$ is a vector bundle of higher rank. The space of effective wave functions of the quasi-hole is a subspace of the vector space $H^0(\Sigma, W_{2,1})$ of global sections of $W_{2,1}$. We can compute the rank and the first Chern class of the vector bundle $W_{2,1}$ where the physical traits of the quasi-hole are encoded.

In order to do this let us consider an isogeny $\varphi : \Sigma \to \Sigma$, which is the finite morphism of degree $(k+1)^2$, such that

$$\varphi^* W_{2,1} = H^0(\Sigma, \mathcal{L}_{k+1})^{\otimes 2} \otimes \mathcal{L}_{2k(k+1)}. \tag{30}$$

Proof of the existence of such a morphism will appear elsewhere [14] and depends on study of the Picard bundles over abelian varieties [16]. From the identity (30) one can deduce that

$$\operatorname{rank} W_{2,1} = (k+1)^2, \qquad c_1(W_{2,1}) = 2k(k+1)$$

explaining the notation $W_{2,1}$. The slope is

$$\mu(W_{2,1}) = \frac{2k}{k+1} = \frac{4m}{2m+1}.$$

To extract physical information from the geometric formulation we remark that Haldane and Rezayi analysis is based on the study of the set of zeros of $F_l^{\mathrm{QH}}(z_0, z_1, z_2)$. The number of zeros of any section in $H^0(\Sigma, W_{2,1})$ is lower than or equal to the degree of $W_{2,1}$. The vector bundle $\varphi^* W_{2,1}$ takes into account this fact together with the morphism ξ_{12}^*, which pulls back from the CM and relative coordinates of the two electrons. The rank of $\varphi^* W_{2,1}$ is the same as the rank of $W_{2,1}$ but the degree, due to the finite morphism, is $c_1(\varphi^* W_{2,1}) = 2k(k+1)^3$. The slope

$$\mu(\varphi^* W_{2,1}) = 4m(k+1) \tag{31}$$

encodes the charge of the quasi-hole via a generalization of Dirac's quantization condition:

$$\mu(\varphi^* W_{2,1}) = q_{\mathrm{H}} \Phi_{\mathrm{H}}. \tag{32}$$

But $\Phi_{\mathrm{H}} = 4m^2(k+1)$, so that

$$q_{\mathrm{H}} k^2 (k+1) = 4m(k+1). \tag{33}$$

On the left-hand side of (33) we have the product of the charge of the quasi-hole times the total magnetic flux which picks up three multiplicative contributions: (1) $(k+1)$ is the magnetic flux in the unit principal cell due to the external magnetic field and the QH. (2)

m^2 comes from the scale factor of the area which arises because the QH coordinate z_0 is $1/m$ times the particle coordinates z_i. (3) Finally factor 4 is due to the process of taking CM and relative coordinates. The right-hand member is a topological invariant, the slope of the $\varphi^* W_{2,1}$ bundle, implying that $q_{\mathrm{H}} = 1/m$. This agrees with the count of zeros of F_l^{QH} which is $m(k+1) = \frac{1}{4}\mu(\varphi^* W_{2,1})$.

3.3.1.2. Two electrons and two quasi-holes. Let us consider the morphisms:

$$\begin{array}{ccccc} & & \Sigma & & \Sigma \\ & & s_{14}\nwarrow & & \nearrow s_{24} \\ \Sigma \xleftarrow{s_{34}} \Sigma^{(4)} & = & \Sigma \times \Sigma \times \Sigma \times & \Sigma & \xrightarrow{\xi_{12}} \Sigma \times \Sigma \\ & & s_{13}\swarrow & & \searrow s_{23} \\ & & \Sigma & & \Sigma \end{array}$$

defined by

$$\begin{aligned} &\xi_{12}(z_1, z_2, z, w) = (z_1 + z_2, z_1 - z_2), \\ &s_{13}(z_1, z_2, z, w) = z_1 - z, \qquad s_{14}(z_1, z_2, z, w) = z_1 - w, \\ &s_{23}(z_1, z_2, z, w) = z_2 - z, \qquad s_{24}(z_1, z_2, z, w) = z_2 - w, \\ &s_{34}(z_1, z_2, z, w) = z - w. \end{aligned}$$

Under the condition $k = 2m$ the line bundle over $\Sigma^{(4)}$ describing the system is

$$\begin{aligned} M_{2,2} = {}& \xi_{12}^*(p_1^* \mathcal{L}_m \otimes p_2^* \mathcal{L}_m) \otimes s_{13}^* \mathcal{L}_1 \otimes s_{23}^* \mathcal{L}_1 \\ & \otimes s_{14}^* \mathcal{L}_1 \otimes s_{24}^* \mathcal{L}_1 \otimes s_{34}^* \mathcal{L}_1. \end{aligned}$$

Under the change of variables $\sigma : \Sigma^{(4)} \to \Sigma^{(4)}$, $\sigma(z_1, z_2, z, w) = (z_1, z_2, -z, -w)$ the bundle $M_{2,2}$ becomes

$$\begin{aligned} N_{2,2} = \sigma^* M_{2,2} = {}& p_1^* \mathcal{L}_k \otimes p_2^* \mathcal{L}_k \otimes m_{13}^* \mathcal{L}_1 \otimes m_{14}^* \mathcal{L}_1 \\ & \otimes m_{23}^* \mathcal{L}_1 \otimes m_{24}^* \mathcal{L}_1 \otimes \bar{s}_{34}^* \mathcal{L}_1, \end{aligned}$$

where $\bar{s}_{34}(z, w) = w - z$ and m_{ij} are the partial addition maps. The quantum space of states of two electrons and two holes is the subspace $E_k(2,2)$ of the space of holomorphic sections of $N_{2,2}$, $H^0(\Sigma^{(4)}, N_{2,2})$, if $V_{1/m} = H^0(\Sigma, \mathcal{L}_1)$.

Again there are distinguished wave functions for two quasi-holes:

$$\begin{aligned} F_l^{2\mathrm{QH}} = {}& \Theta \begin{bmatrix} l/m \\ 0 \end{bmatrix} (m(z_1 + z_2) + z + w | m\tau) \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_1 - z_2 | \tau) \\ & \times \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_1 - z | \tau) \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_1 - w) \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_2 - z | \tau) \\ & \times \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_2 - w | \tau) \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z - w | \tau), \end{aligned} \tag{34}$$

which is a section in $E_k(2,2)$ with no zeros away from the diagonal.

The bundle of effective wave functions of the two quasi-holes is the bundle over $\Sigma \times \Sigma$ defined by $\Pi_{34}^* N_{2,2} = W_{2,2}$, where $\Pi_{34}: \Sigma^{(4)} \to \Sigma \times \Sigma$ is the natural projection on the last two factors. Let us observe that $W_{2,2}$ can be interpreted as the bundle over $\Sigma \times \Sigma$ obtained by integrating $N_{2,2}$ with respect to the first two variables. The vector space of effective wave functions of two quasi-holes is therefore $H^0(\Sigma \times \Sigma, W_{2,2})$, the space of global sections of $W_{2,2}$. The rank of $W_{2,2}$ is $(k+2)^2$.

The effective wave function for the selected two quasi-hole wave function above is proportional to

$$\Theta^{1/m} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z - w|\tau).$$

Exchanging two quasi-holes amounts to changing $z - w$ to $w - z$; a factor $\mathrm{e}^{\mathrm{i}\pi/m}$ arises and the statistics of the quasi-holes is anyonic, in keeping with the fact that their electric charge is fractionary (see [17] for a similar argument in the complex plane).

3.3.1.3. N electrons and one quasi-hole. The bundle characterizing the system is a line bundle over $\Sigma_{(0)} \times (\Sigma_{(1)} \times \cdots \times (N\text{times})\Sigma_{(N)})$ of the form:

$$M_{N,1} = (p_1^* \mathcal{L}_k \otimes \cdots \otimes p_N^* \mathcal{L}_k) \otimes (\otimes_{i=1}^N s_{i0}^* \mathcal{L}_1),$$

where s_{i0} is the morphism $\Sigma_{(0)} \times \Sigma_{(i)} \to \Sigma$ defined by $s_{i0}(z_0, z_i) = z_i - z_0$. As in the N-electron case one considers the morphism $\varphi_N : \Sigma^N \to \Sigma^N$, $\varphi_N(z_1, z_2, \cdots, z_N) = (\sum_{i=1}^N z_i, z_1 - z_2, \cdots, z_1 - z_N)$ to identify $M_{N,1}$ with

$$M_{N,1} = \varphi_N^*(p_1^* \mathcal{L}_m \otimes \cdots \otimes p_N^* \mathcal{L}_m) \otimes \left(\bigotimes_{i=1}^N s_{i0}^* \mathcal{L}_1 \right),$$

where $k = Nm$. It is also convenient to use another morphism $\sigma : \Sigma \times \Sigma^N \to \Sigma \times \Sigma^N$, $\sigma(z_0, z_1, \ldots, z_N) = (-z_0, z_1, \ldots, z_N)$, to obtain the bundle

$$N_{N,1} = \sigma^* M_{N,1} = (p_1^* \mathcal{L}_k \otimes \cdots \otimes p_N^* \mathcal{L}_k) \otimes \left(\bigotimes_{i=1}^N m_{i0}^* \mathcal{L}_1 \right).$$

The quantum space of states is a subspace of the space of global sections $H^0(\Sigma \times \Sigma^N, N_{N,1})$ of $N_{N,1}$. A wave function for the quasi-hole is a section in $E_k(N, 1)$

$$F_l^{\mathrm{QH}}(z_0, z_1, \ldots, \_N) = \Theta \begin{bmatrix} l/m \\ 0 \end{bmatrix} \left(z_0 + \sum_{i=1}^N z_i | m\tau \right) \prod_{i<j} \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_i - z_j|\tau)$$
$$\times \prod_{i=1}^N \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_i - z_0|\tau)$$

distinguished by having no zeros away from the diagonals.

The bundle of effective wave functions of the quasi-hole is obtained by integrating $N_{N,1}$ along the variables $z_1, \ldots, z_N$:

$$W_{N,1} = p_{0*}N_{N,1} = p_{0*}\bigotimes_{i=1}^{N}(p_i^*\mathcal{L}_k \otimes m_{i0}^*\mathcal{L}_1),$$

where $p_0 : \Sigma \times \Sigma^N \to \Sigma$ is the projection on the first factor. The vector space of effective wave functions of the quasi-hole is precisely $H^0(\Sigma, W_{N,1})$. We have for the rank, first Chern class and slope of $W_{N,1}$ the following:

$$\operatorname{rank} W_{N,1} = (k+1)^N, \qquad c_1(W_{N,1}) = Nk(k+1)^{N-1}, \qquad \mu(W_{N,1}) = \frac{Nk}{k+1}.$$

The Dirac quantization condition (33) becomes

$$q_{\mathrm{H}}k^2(k+1) = N^2m(k+1), \tag{35}$$

and the electric charge of the quasi-hole is $q_{\mathrm{H}} = 1/m$. To derive (35), a parallel analysis to that previously performed leading to (33) should be done; again, there is an isogeny $\varphi : \Sigma_0 \to \Sigma_0$ of degree $(k+1)^2$ and rank $(\varphi^* W_{N,1}) = (k+1)^N, c_1(\varphi^* W_{N,1}) = Nk(k+1)^{N+1}$. The slope

$$\mu(\varphi^* W_{N,1}) = N^2m(k+1)$$

is the right hand side of (35) which encodes the electric charge of quasi-holes by an identical physical analysis that is used in the discussion developed in the $N = 2$ electron case.

3.3.2. Geometric formulation of a system with quasi-particles

Finally, let us rigorously define a system with two electrons and one quasi-particle moving in a periodic lattice. In this case we arrive at a surprising conclusion in some sense hinted by Haldane and Rezayi: there are no quantum states for the effective wave function of one quasi-particle. This phenomenon is reminiscent of what happens in the complex plane. Therefore, the definition of quasi-particle requires that some states should jump to the next Landau level. The loophole is to interpret the non-analytic states as the annihilation of some honest states in the first Landau level that does not enter the quasi-particle wave function. This is not possible in the case of a periodic lattice; the isogenies of degree higher than 1 involved force all the states into higher Landau levels. There are no global sections and no effective wave functions for quasi-particles on a torus.

As in Sections 3.2 and 3.3 consider the morphisms $\Sigma \times \Sigma \times \Sigma \underset{s_{20}}{\overset{s_{10}}{\rightrightarrows}} \Sigma$ defined by: $s_{10}(z_0, z_1, z_2) = z_1 - z_0, s_{20}(z_0, z_1, z_2) = z_2 - z_0$ as well as $\xi_{12} : \Sigma \times \Sigma \times \Sigma \to \Sigma \times \Sigma$ defined in Section 3.3.1.1. The system of two electrons and one quasi-particle is defined by the line bundle over $\Sigma \times \Sigma \times \Sigma$:

$$\begin{aligned} M_2^1 &= \xi_{12}^*(p_1^*\mathcal{L}_m \otimes p_2^*\mathcal{L}_m) \otimes s_{10}^*\mathcal{L}_1^{-1} \otimes s_{20}^*\mathcal{L}_1^{-1} \\ &= (p_1^*\mathcal{L}_k \otimes p_2^*\mathcal{L}_k) \otimes s_{10}^*\mathcal{L}_1^{-1} \otimes s_{20}^*\mathcal{L}_1^{-1}, \end{aligned}$$

where $k = 2m$.

By an argument similar to that used in Section 3.3.1.1, we can compute $W_2^1 = p_0 * M_2^1$. There exists in this case an isogeny $\varphi : \Sigma \to \Sigma$ of degree $(k-1)^2$ such that

$$\varphi^*(W_2^1) = H^0(\Sigma, \mathcal{L}_{k-1})^2 \otimes \mathcal{L}_1^{\otimes -2k(k-1)}.$$

Then we have that: rank $(W_2^1) = (k-1)^2 =$ rank $(\varphi^* W_2^1)$, $c_1(\varphi^* W_2^1) = -2k(k-1)^3$, $c_1(W_2^1) = -2k(k-1)$. The slopes are: $\mu(W_2^1) = -2k/(k-1)$ and $\mu(\varphi^* W_2^1) = -2k(k-1)$.

Obviously $H^0(\Sigma, \varphi^* W_2^1) = 0$. From this fact one easily deduces that $H^0(\Sigma, W_2^1) = 0$; the vector space of the effective wave functions of the quasi-particle in a periodic lattice is empty.

In a physical vein this means that the Haldane–Rezayi ansatz

$$F_l^{\mathrm{QP}}(z_0, z_1, \ldots, z_N) = \prod_{i=1}^{N} \Theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \left(2\frac{\partial}{\partial z_i} - z_0\right) \Theta \begin{bmatrix} l/m \\ 0 \end{bmatrix} \left(z_0 + m \sum_{i=1}^{N} z_i | m\tau\right) \times \prod_{j<l} \Theta^m \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_j - z_l | \tau)$$

envisaged by them as the likely wave function for quasi-particles is not holomorphic because of the existence of poles.

4. Vector bundles over elliptic curves

In this section we analyze how the Hall conductivity is defined as the slope of certain vector bundles related to the center of mass dynamics in the FQHE. Extension of the Thouless approach [5] to the fractionary case requires remarkable modifications and leads to a very subtle description of the plateaux, which in samples with weak impurities is due to another physical mechanism.

4.1. The Hall conductance as a topological invariant

A synthesis of the Thouless formulation runs as follows: Connect two current leads to the two-dimensional electronic bar, see Fig. 1, in order to study variations in j induced by solenoid fluxes $\phi(t)$ passing through the current loops and slowly changing in time. Due to the Aharonov–Bohm effect, the electron wave functions feel global phase factors if $\phi_x, \phi_y \in [0, e/hc]$ and the physics is periodic in ϕ with e/hc period.

The many-electron Hilbert space is the fermionic Fock space [3] obtained from $L^2(\mathbb{C})$, the one-particle space of states, by anti-symmetric tensor product: if $f(z, \bar{z}) \in L^2(\mathbb{C})$, an element $\Psi \in \mathcal{F}_-^{(N)}(L^2(\mathbb{C}))$ has of the form:

$$\Psi = P_-^N f_1 \otimes f_2 \otimes \cdots \otimes f_N = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{P(\sigma)} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(N)}.$$

$p(\sigma) = \pm 1$ is the parity of an element σ of the permutation group S_N.

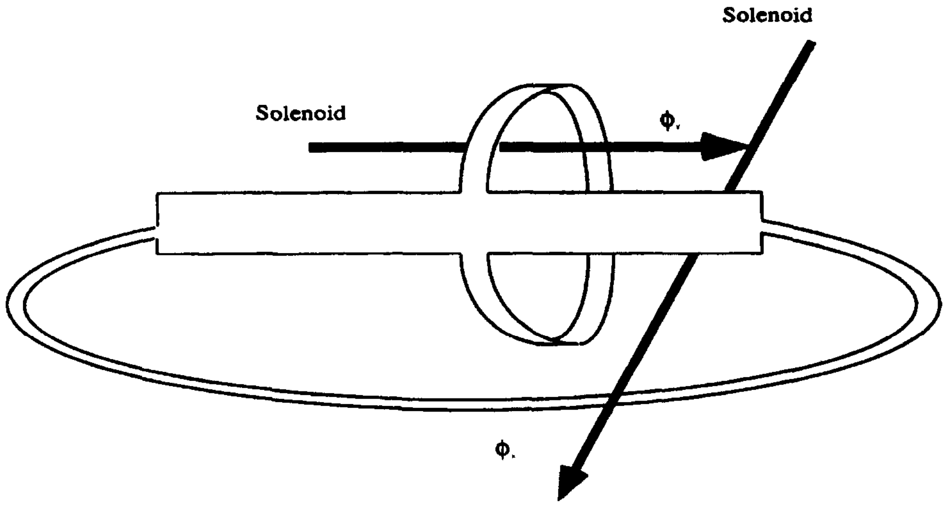

Fig. 1. The double Aharanov–Bohm experiment.

Formally, the fermionic Fock space,

$$\mathcal{F}_-(L^2(\mathbb{C})) = \sum_{N=0}^{\infty} \mathcal{F}_-^{(N)}(L^2(\mathbb{C})) = \sum_{N=0}^{\infty} \Lambda^N L^2(\mathbb{C}),$$

is the exterior algebra of $L^2(\mathbb{C})$. From the skew symmetric product

$$\langle g|f\rangle = \int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i} g^*(z,\bar{z}) f(z,\bar{z}) = \langle f|g\rangle^*$$

an L^2-structure in $\mathcal{F}_-^{(N)}(\mathbb{C})$ is induced:

$$\langle \Psi|\Phi\rangle = \int \prod_{i=1}^{N} \frac{\mathrm{d}\bar{z}_i \mathrm{d}z_i}{2\mathrm{i}} \Psi^*(z_1,\bar{z}_1;\ldots;z_N,\bar{z}_N)\Phi(z_1,\bar{z}_1;\ldots;z_N,\bar{z}_N).$$

The many-body Hamiltonian describing the dynamics of the two-dimensional electron system is

$$H(\phi(t),\bar{\phi}(t)) = \sum_{i=1}^{N} \left\{ \frac{1}{2m_e}(|\partial_A^{(i)} + \phi(t)|^2) + U_1(z^{(i)},\bar{z}^{(i)}) \right\} + \sum_{i<j} U_2(z^{(i)},\bar{z}^{(i)}; z^{(j)},\bar{z}^{(j)}).$$

Here $\phi(t) = \phi_x(t) + \mathrm{i}\phi_y(t)$, $\partial_A^{(i)} = \partial_{z^{(i)}} + \frac{1}{2}eB\bar{z}^{(i)}$, $z^{(i)} = x^{(i)} + \mathrm{i}y^{(i)}$ is the coordinate of the ith electron. We choose $\boldsymbol{B} = B\boldsymbol{k}$, $-e$ is the electron charge and U_1, U_2 are the one- and two-body energy interactions for the electrons, respectively.

The Kubo–Thouless formula computes the Hall conductance in first-order perturbation theory induced by adiabatic changes in $\phi(t)$:

$$\sigma_{\mathrm{H}} = \mathrm{i}\hbar c^2 \left[\left\langle \frac{\partial \Psi_0}{\partial \phi_x} \middle| \frac{\partial \Psi_0}{\partial \phi_y} \right\rangle - \left\langle \frac{\partial \Psi_0}{\partial \phi_y} \middle| \frac{\partial \Psi_0}{\partial \phi_x} \right\rangle \right]$$

if Ψ_0 is the ground state of the system, the eigenstate of lowest eigenvalue of $H(\phi(0), \bar{\phi}(0))$.

Physical arguments show that σ_{H} is a topological invariant independent of time: impurities are taken into account in the Hamiltonian by the U_{I} interaction energy and they create localized states which do not contribute to the Hall current to be added to the extended Landau states. Adiabatic changes in the fluxes trigger changes in ν_{H} but not in conductance, the injected electrons filling in localized states.The non-zero matrix elements $\langle \Psi_0 | \partial H / \partial \phi | \Psi_E \rangle$ occur only for values of the energy in a mobility gap. However, because the fluxes change in time σ_{H} is also independent of $\phi(t)$ and hence, equal to its average value:

$$\langle \sigma_{\mathrm{H}} \rangle = \frac{1}{2\pi \mathrm{i}} \frac{e^2}{h} \int_0^1 \mathrm{d}\phi_1 \int_0^1 \mathrm{d}\phi_2 \left[\left\langle \frac{\partial \Psi_0}{\partial \phi_1} \middle| \frac{\partial \Psi_0}{\partial \phi_2} \right\rangle - \left\langle \frac{\partial \Psi_0}{\partial \phi_2} \middle| \frac{\partial \Psi_0}{\partial \phi_1} \right\rangle \right], \tag{36}$$

where $\phi_x = (hc/e)\phi_1$, $\phi_y = (hc/e)\phi_2$.

Complex line bundles $\mathcal{L}(\Sigma; \mathbb{C})$ over a genus 1 Riemann surface are topologically classified by the first Chern class. Via Chern–Weyl theory it is determined in terms of the curvature of a $U(1)$-connection: $c_1(\mathcal{L}) = (1/2\pi) \int_\Sigma F_\alpha$. If Σ is the torus parametrized by ϕ_1, ϕ_2 we see that $\sigma_{\mathrm{H}} = (e^2/h) c_1(\mathcal{L})$, the Hall conductance is a topological invariant because the integrand in (36) is

$$\mathrm{d}\alpha = -2\mathrm{i}\,\mathrm{Im} \left\langle \frac{\partial \Psi_0}{\partial \phi_1} \middle| \frac{\partial \Psi_0}{\partial \phi_2} \right\rangle \mathrm{d}\phi_1 \mathrm{d}\phi_2, \tag{37}$$

where $\alpha = -2\mathrm{i}\,\mathrm{Im}\, \langle \Psi_0 | \partial \Psi_0 / \partial \phi_2 \rangle\, \mathrm{d}\phi_2$. The fact that $\langle \sigma_{\mathrm{H}} \rangle = \sigma_{\mathrm{H}}$, of physical origin, geometrically means that the curvature tensor F_α is constant in Σ.

Therefore, $\sigma_{\mathrm{H}} = (e^2/h)n$, $n \in \mathbb{Z}$, because any possible connection in (37) is topologically equivalent to $\alpha^{(n)} = -2\pi \mathrm{i} \phi_1 \mathrm{d}\phi_2$, independently of the details of the ground state Ψ_0. The quantification of the Hall conductance in the QHE, as well as the existence of plateaux (only integer jumps are allowed), is explained in this way by recognizing σ_{H} as a characteristic class.

One might try to apply the same analysis to the fractional quantum Hall effect looking for a ground state r times degenerated. The Kubo–Thouless formula

$$\sigma_{\mathrm{H}} = \frac{e^2}{r} \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{r} \int_0^1 \mathrm{d}\phi_1 \int_0^1 \mathrm{d}\phi_2 \left[-2\mathrm{i}\,\mathrm{Im} \left\langle \frac{\partial \Psi_0^i}{\partial \phi_1} \middle| \frac{\partial \Psi_0^i}{\partial \phi_2} \right\rangle \right]$$

identifies σ_{H} as the slope of a rank r complex vector bundle over Σ, $E(\Sigma; \mathbb{C}^r)$; $\sigma_{\mathrm{H}} = \mu(E) e^2/h = (c_1(E)/r) e^2/h$. But an important problem arises: for the FQHE there is a very sound variational ground state proposed by Laughlin [4], which is non-degenerated.

To explain the existence of plateaux in the FQHE another point of view must be explored.

There is a refinement of the Laughlin theory of interest for understanding $\sigma_{\rm H}$ in the FQHE as a topological quantity: when the FQHE occurs in a periodic lattice there is a variational ground state of Laughlin type [10], that we have studied in the previous section. The most important point is that this ground state is degenerated due to the dynamics of the center of mass. This fact has been brilliantly exploited by Varnhagen [11], to identify $\sigma_{\rm H}$ as the slope of certain vector bundles constructed from the center of mass contribution to the Haldane–Rezayi wave function.

4.2. Center of mass dynamics

Center of mass degeneracy is crucial in the intelligence of Hall conductance as a topological invariant in the FQHE; it is thus important to unveil what the dynamical origin of this degeneracy is. We shall recall how it comes about from an abelian topological field theory of Chern–Simons type, along the lines developed in [18]. The periodicity conditions, $Z = \sum_{i=1}^{N} z_i$,

$$F_{\rm cm}(Z+1) = {\rm e}^{2\pi {\rm i}\phi_1} F_{\rm cm}(Z),$$
$$F_{\rm cm}(Z+\tau) = {\rm e}^{-2\pi {\rm i}\phi_2} {\rm e}^{-2\pi {\rm i} m(Z+\tau/2)} F_{\rm cm}(Z) \tag{38}$$

satisfied by the basis states

$$F_{\rm cm}(\phi_1, \phi_2; Z) = \Theta \begin{bmatrix} (l+\phi_1)/m \\ \phi_2 \end{bmatrix} (mZ|m\tau)$$

of $H^0(\Sigma, \mathcal{L}_m)$, have their physical roots in the periodic lattice and the inclusion of the magnetic fluxes $\phi_1 = (e/hc)\phi_{x_1}, \phi_2 = (e/hc)\phi_{x_2} \in [0, 1]$, in the double Aharanov–Bohm arrangement described in Section 1. Mathematically, $\phi = \phi_1 + {\rm i}\phi_2$ is a local coordinate in $\hat{\Sigma}$, the dual to the original curve, because it parametrizes the manifold of complex flat line bundles over Σ. $\hat{\Sigma}$ is thus the Jacobian or Picard group of Σ, which is isomorphic to Σ itself.

Consider a family of $U(1)$ connections parametrized by $\mathbb{R}$ on the "first Brillouin zone" $\hat{\Sigma}$:

$$a(t) = a_{\phi_1}(\phi_1, \phi_2; t)\, {\rm d}\phi_1 + a_{\phi_2}(\phi_1, \phi_2; t){\rm d}\phi_2. \tag{39}$$

The gauge group $\mathcal{G} = {\rm Maps}(\hat{\Sigma}, U(1))$ acts on $a(t)$ in an affine way:

$$g\dot{a} = a + {\rm i}g{\rm d}g^{-1} = a + {\rm d}\omega$$

if $g(\phi_1, \phi_2) = \exp({\rm i}\omega(\phi_1, \phi_2)) \in \mathcal{G}$.

The Chern–Simons action is

$$S = \frac{m}{2} \int\limits_{\hat{\Sigma}\times\mathbb{R}} a(t) \wedge {\rm d}^{(3)} a(t), \tag{40}$$

where $\mathrm{d}^{(3)} = \mathrm{d} + \partial_t t$. Classically, there is a dynamical equation and a constraint equation, the Gauss law:

$$\frac{\partial a}{\partial t} = 0, \qquad * \mathrm{d}a = 0. \tag{41}$$

To quantize the system we take the quotient by the gauge group imposing the Coulomb gauge, $\mathrm{d}^* a = 0$, and reduce the phase space by solving the constraint. The symplectically reduced phase space is the moduli space of the gauge equivalent flat connections, $F_{a_f} = 0$,

$$a_f = c_1\, \mathrm{d}\phi_1 + c_2\, \mathrm{d}\phi_2$$

with action

$$S_R[a_f(t)] = \frac{m}{2\,\mathrm{Im}\,\tau} \int t(c_1 \dot{c}_2 - c_2 \dot{c}_1). \tag{42}$$

We have imposed periodicity on the ϕ_1- and ϕ_2-axis and have encoded the modular parameter in the complex structure. The c_a are thus independent of ϕ_1, ϕ_2 and take values in [0, 1].

A choice of polarization in the reduced phase space $\hat{\Sigma} = H^1(\Sigma, U(1))$, taking c_1 as the "coordinates", and c_2, as the "momentum", paves the way to geometric quantization: the quantum space of states of the system is the $L^2(\mathbb{R})$ Hilbert space of functions satisfying

$$\Psi(c_1 + e) = \Psi(c_1), \qquad \Psi(c_1)\mathrm{e}^{2\pi \mathrm{i} m c_1} = \Psi(c_1). \tag{43}$$

A basis in the Hilbert space is formed by the eigenfunctions of the "position" operator in $\hat{c}_1$:

$$\Psi(c_1) = \sum_{n \in \mathbb{Z}} \delta\left(c_1 + \frac{l}{m} - n\right)$$

$l \in \mathbb{Z}_m$, with eigenvalue l/m.

The dimension of the space of states is thus m. Another basis is given by the Landau states, eigenfunctions of the operator $\bar{\partial}^a_c = \partial_{c_1} + \mathrm{i}\partial_{c_2} + \mathrm{i}(2\pi m/\mathrm{Im}\,\tau)c_2$, satisfying boundary conditions as (38), of zero eigenvalue:

$$\phi_l(c|\tau) = \Theta \begin{bmatrix} (l + \phi_1)/m \\ \phi_2 \end{bmatrix} (mc|m\tau).$$

Recognizing the center of mass wave functions is immediate: simply identify c as Z!. The outcome is that the center of mass dynamics is governed by an abelian Chern–Simons theory over the first magnetic Brillouin zone.

There is more information in the space of states describing the CM dynamics. A rank m vector bundle over $\hat{\Sigma}$ can be construted along similar lines to that of the papers by Varnhagen [11] and Novikov [19]. First observe that

$$f_l(\phi_1, \phi_2) = \Theta \begin{bmatrix} l/m \\ 0 \end{bmatrix} (m\eta|m\tau), \qquad \eta = \frac{\phi_1}{m}\tau + \phi_2.$$

E_m is the vector bundle over $\hat{\Sigma}$ whose fiber over (ϕ_1, ϕ_2) is the vector space generated by $\{f_1, f_2, \ldots, f_m\}$. The principal cell $\phi_1, \phi_2 \in [0, 1]$ in terms of η is $\eta_1, \eta_2 \in [0, 1/m]$.

The periodicity $\phi_1 \to \phi_1 + 1$ goes to $\eta_1 \to \eta_1 + \tau/m$ and $\phi_2 \to \phi_2 + 1$ becomes $\eta_2 \to \eta_2 + 1$. There is a single zero in the principal cell of $f_{m-1}(\phi)$, $\eta^{(0)} = (1/2m)\tau + 1/2m$, or, $\phi_1^{(0)} = \phi_2^{(0)} = 0$, and no zeros at all for the others f_l. This property is related, as we shall see in the next sections, to the first Chern class of the E_m vector bundle.

4.3. Vector bundles over magnetic Brillouin zones

The aim of this section is to provide a rigorous geometrical construction of the higher rank vector bundles introduced above in connection with the center of mass dynamics. Recall that $\hat{\Sigma}$ is the moduli space of vector bundles over Σ of 0 degree and rank 1. Let $\mathcal{P}$ be the Poincaré sheaf over $\Sigma \times \hat{\Sigma}$: $\mathcal{P}$ is a line bundle whose fiber over a point $[\phi] \in \hat{\Sigma}$ defining a line bundle $\mathcal{L}_0(\phi)$ over Σ is precisely $\mathcal{L}_0(\phi)$:

$$\mathcal{P} \mid_{\Sigma \times [\phi]} = \mathcal{L}_0(\phi).$$

Let π_Σ and $\pi_{\hat{\Sigma}}$ be, respectively, the two natural projections of $\Sigma \times \hat{\Sigma}$ into Σ and $\hat{\Sigma}$. The physical situation in which one electron moves on a torus in the presence of a constant magnetic field of total flux $2\pi k$ and solenoid fluxes ϕ_1, ϕ_2 arranged as in the Thouless gedanken experiment is set forth in mathematical terms by the line bundle over $\Sigma \times \hat{\Sigma}$:

$$L_k = \pi_\Sigma^* \mathcal{L}_k \otimes \mathcal{P},$$

where $\mathcal{L}_k = \theta_\Sigma(ke)$ is the line bundle of kth-order Theta functions over Σ. The fiber of L_k in a point $[\phi] \in \hat{\Sigma}$ is

$$L_k \mid_{\Sigma \times [\phi]} = (\pi_\Sigma^* \mathcal{L}_k \otimes \mathcal{P})_{\Sigma \times [\phi]} = \mathcal{L}_k \otimes \mathcal{L}_0(\phi)$$

and we can interpret L_k as the line bundle over $\Sigma \times \hat{\Sigma}$ which parametrizes all the non-equivalent vector potentials leading to magnetic fields with total magnetic flux through Σ equal to $2\pi k$, k being the first Chern class. Another mathematical construction of physical interest is the Fourier–Mukai transform of $\mathcal{L}_k$, $W(\mathcal{L}_k)$: $W(\mathcal{L}_k)$ is a rank k vector bundle over $\hat{\Sigma}$ which is obtained by taking the direct image to the last factor $\hat{\Sigma}$,

$$W(\mathcal{L}_k) = \pi_{\hat{\Sigma}*}(\pi_\Sigma^* \mathcal{L}_k \otimes \mathcal{P}).$$

$W(\mathcal{L}_k)$ represents the dependence of L_k in the last variable and its fiber at a point $[\phi] \in \hat{\Sigma}$ is $H^0(\Sigma, \mathcal{L}_k \otimes \mathcal{L}_0(\phi))$.

Let us consider a system of N electrons in a periodic lattice under magnetic fields as described above. Recall that basic line bundle over $(\Sigma \times \cdots \times \Sigma)$ (N times) for N electrons in a Hall fluid is: if $k = mN$,

$$M_N = m_N^* \mathcal{L}_m \left(\bigotimes_{\substack{i<j \\ i \geq 1}}^{N} s_{ij}^* \mathcal{L}_m \right),$$

where m_N is the total addition morphism, s_{ij} are the subtraction morphisms of Section 3.2, and $\mathcal{L}_m = \theta_\Sigma(me)$. Quantum fluctuations suppress the dependence on the solenoid fluxes of the contribution to the wave functions of relative coordinates z_{ij}.Only the CM quantum dynamics is affected by the solenoids and depends on ϕ. Therefore, the line bundle $\mathcal{M}_N$ over $\hat{\Sigma} \times \Sigma \times \cdots \times \Sigma(N\text{times})$ describing this more general situation is

$$\mathcal{M}_N = m_N^* L_m \otimes \left(\bigotimes_{\substack{i<j \\ i=1}}^{N} s_{ij}^* \mathcal{L}_m \right),$$

where L_m includes the effect of the solenoid fluxes on the CM dynamics by means of the Poincaré bundle construction:

$$L_m = \pi_\Sigma^* \mathcal{L}_m \otimes \mathcal{P}.$$

In fact, following Varnhagen, we can consider exclusively the CM contribution. Geometrically, this amounts to focusing on the dependence of the bundle L_m on the first variable; the bundle

$$W_N(\mathcal{L}_m) = \pi_{\hat{\Sigma}*}(m_N^* L_m)$$

is therefore the bundle defined by (4.1), (4.2) and (4.3) in [11] and referred to as E_m in Section 4.1.

It is interesting to see that the bundle $W_N(\mathcal{L}_m)$ describing the dynamics of the center of mass is in a certain sense the Fourier–Mukai transform of a line bundle. Let us denote $Z = (\Sigma \times \cdots \times \Sigma(N\text{times})$ and $\hat{Z} = (\hat{\Sigma} \times \cdots \times \hat{\Sigma}(N\text{times})$ for short and let Π_Z and $\Pi_{\hat{Z}}$ be the two natural projections of $Z \times \hat{Z}$ into Z and $\hat{Z}$, respectively. Let I_m be the line bundle $(p_1 + \cdots + p_N)^* \mathcal{L}_m$ over $Z = \Sigma \times \cdots \times \Sigma$, $p_i : Z \to \Sigma$ being the natural projections. The Fourier–Mukai transform of the line bundle I_m is the bundle over $\hat{Z}$ given by

$$\Omega_N(I_m) = \Pi_{\hat{Z}*}(\Pi_Z^* I_m \otimes \mathcal{P}_Z),$$

where $\mathcal{P}_Z$ is the Poincaré bundle over $Z \times \hat{Z}$ (see [16,17] for details).

If we denote by $\hat{\Sigma} \xrightarrow{\Delta} \hat{Z}$ the diagonal immersion, $\Delta(\phi) = (\phi, \phi, \ldots, \phi)$, our bundle $W_N(\mathcal{L}_m)$ is related to the Fourier–Mukai bundle $\Omega_N(I_m)$ by the formula

$$W_N(\mathcal{L}_m) = \Delta^* \Omega_N(I_m) = \Delta^* \Pi_{\hat{Z}*}(\Pi_Z^* I_m \otimes \mathcal{P}_Z).$$

One deduces that $W_N(\mathcal{L}_m)$ is a vector bundle over $\hat{\Sigma}$ of the same rank as the vector bundle $\Omega_N(I_m)$ over $\hat{Z}$. But the rank of $\Omega_N(I_m)$ can be easily computed (see [16, Lemma 1])

$$\operatorname{rank} \Omega_N(I_m) = \dim H^0(Z, I_m) = \dim H^0(\Sigma, \mathcal{L}_m) = m$$

in keeping with the physicist's description in Section 4.2.

To compute the degree of $W_N(\mathcal{L}_m)$ consider again the isomorphism $\xi : Z \to Z$, $\xi(z_1, z_2, \ldots, z_N) = (z_1, z_2 - z_1, z_3, \ldots, z_N)$. Then

$$\xi^*(m_N^*(\Pi_\Sigma^* \mathcal{L}_m \otimes \mathcal{P})) \cong \Pi_{N-1}{}^*(m_{N-1}^*(\Pi_\Sigma^* \mathcal{L}_m \otimes \mathcal{P})),$$

where $\Pi_{N-1} : \Sigma_{(1)} \times \cdots \times \Sigma_{(N)} \to \Sigma_{(2)} \times \cdots \times \Sigma_N$ is the natural projection onto the last $N-1$ factors.

There is an isomorphism between vector bundles:

$$\Pi_{\hat{\Sigma}*}(\xi^*) : W_N(\mathcal{L}_m) \xrightarrow{\sim} W_{N-1}(\mathcal{L}_m).$$

In some sense the fiber bundles $W_N(\mathcal{L}_m)$ are independent of the number of particles. In particular

$$\deg W_N(\mathcal{L}_m) = \deg W_{N-1}(\mathcal{L}_m) = \deg W_1(\mathcal{L}_m), \quad \forall N \in \mathbb{N}.$$

$W_1(\mathcal{L}_m) = \Pi_{\hat{\Sigma}*}(\Pi_\Sigma^* \mathcal{L}_m \otimes \mathcal{P})$ is the Fourier–Mukai transform of $\mathcal{L}_m$. Let us consider the isogeny $\varphi_\mathcal{L} : \Sigma \to \hat{\Sigma}$ defined by

$$\varphi_\mathcal{L}(z) = T_z^* L_m \otimes L_m^{-1},$$

where $T_z : \Sigma \to \Sigma$ is the translation by z. It has been proved in [16] that there is an isomorphism

$$\varphi_\mathcal{L}^* W_1(\mathcal{L}_m) \simeq H^0(\Sigma, L_m) \otimes L_m^{\otimes -1}. \tag{44}$$

One then has

$$\det \varphi_\mathcal{L}^* W_1(\mathcal{L}_m) = (L_m^{\otimes -1})^{\otimes m} = L_m^{\otimes -m}.$$

But $\varphi_\mathcal{L}$ is a morphism of degree m^2, and hence

$$\deg W_1(\mathcal{L}_m) = \frac{\deg(\det \varphi_\mathcal{L}^* W_1(\mathcal{L}_m))}{m^2} = -1$$

and the slope of the bundle $W_1(\mathcal{L}_m)$ is

$$\mu(W_1(\mathcal{L}_m)) = -\frac{1}{m}.$$

From the isomorphism $W_N(\mathcal{L}_m) \xrightarrow{\sim} W_1(\mathcal{L}_m)$ one deduces that

$$\deg W_N(\mathcal{L}_m) = -1, \qquad \mu(W_N(\mathcal{L}_m)) = -\frac{1}{m}.$$

Formula (44) also allows us to prove the stability of the bundle $W_1(\mathcal{L}_m)$ and, bearing in mind the isomorphisms $W_N(\mathcal{L}_m) \xrightarrow{\sim} W_1(\mathcal{L}_m)$, the stability of $W_N(\mathcal{L}_m)$ for arbitrary N:

Let $\mathcal{F} \subset W_1(\mathcal{L}_m)$ be a subbundle of rank $r < m$ and degree d. The stability of $W_1(\mathcal{L}_m)$ is equivalent to proving that

$$\deg(\mathcal{F}) = d < 0$$

for any subbundle $\mathcal{F}$ because then the slope of any subbundle of $W_1(\mathcal{L}_m)$ is strictly lower than the slope of $W_1(\mathcal{L}_m)$ itself. But realizing that $\deg(\varphi_\mathcal{L}^* \mathcal{F}) = \deg(\varphi_\mathcal{L}^*) \cdot \deg \mathcal{F} = m^2 d$, it suffices to prove that the degree of $\varphi_\mathcal{L}^* \mathcal{F}$ is negative. Let us observe that formula (44) implies that $\det(\varphi_\mathcal{L}^* \mathcal{F})$ is a line subbundle of $\Lambda^r \left(H^0(\Sigma, L_m) \otimes L_m^{\otimes -1}\right)$, which can be developed as a direct sum of very negative line bundles over Σ; from this one can deduce that $\deg \varphi_\mathcal{L}^* \mathcal{F} < 0$ and hence, $d < 0$.

That is, we have stated and explained how to prove the following:

Proposition 2. *The vector bundles $W_N(\mathcal{L}_m)$ describing the dynamics of the center of mass are stable vector bundles for every $N > 0$ and every $m > 0$.*

The details of the proof will appear in [14].

4.4. Physics of the slope

From the analysis of Sections 4.2 and 4.3 a fact with important consequences follows:

The filling factor of the Haldane–Rezayi ground state is a quantity of topological nature, essentially the slope of the vector bundle $\Omega_1(L_m)$,

$$\nu_{\mathrm{HR}} = \frac{1}{m} = |\mu(W_1(\mathcal{L}_m)|. \tag{45}$$

It would seem that the identity $\nu_{\mathrm{HR}} = |\mu(W_1(\mathcal{L}_m)|$ is somehow accidental; we have deduced that $\nu_{\mathrm{HR}} = 1/m$ and $|\mu(W_1(\mathcal{L}_m)| = 1/m$ by distinct arguments in completely different contexts.

To establish the connection, we apply the Thouless formula to the HR wave function:

$$\begin{aligned} \sigma_{\mathrm{H}} &= \frac{1}{2\pi \mathrm{i}}\frac{e^2}{hm}\sum_{l=1}^{m}\left[\left\langle \frac{\partial \Psi_l}{\partial \phi_1}\middle| \frac{\partial \Psi_l}{\partial \phi_2}\right\rangle - \left\langle \frac{\partial \Psi_l}{\partial \phi_2}\middle| \frac{\partial \Psi_l}{\partial \phi_1}\right\rangle\right], \\ \langle \sigma_{\mathrm{H}}\rangle &= \left| \int_0^1 \mathrm{d}\phi_1 \int_0^1 \mathrm{d}\phi_2\, \sigma_{\mathrm{H}}(\phi_1,\phi_2)\right|, \end{aligned} \tag{46}$$

where Ψ_l, $l = 1, 2, \ldots, m$, is

$$\begin{aligned} \Psi_l(\phi_1,\phi_2; z_1,\ldots,z_N) &= F_l(\phi_1,\phi_2; z_1,\ldots,z_N)\mathrm{e}^{-(1/2)\sum_i |z_i|^2} \\ &= \Theta\begin{bmatrix} (l+\phi_1)/m \\ \phi_2 \end{bmatrix}\left(m\sum_i z_i | m\tau\right)\prod_{i<j}\Theta\begin{bmatrix} 1/2 \\ 1/2\end{bmatrix}(z_i - z_j|\tau)\mathrm{e}^{-1/2\sum_i |z_i|^2}. \end{aligned}$$

Let $H^\circ(\hat{\Sigma}\times\Sigma\times\cdots\times\Sigma, \mathcal{M}_N)$ be the space of holomorphic sections of the bundle $\mathcal{M}_N$ defined in the Section 4.2. Let us consider the subspace $h^\circ(\hat{\Sigma}\times\Sigma\times\cdots\times\Sigma, \mathcal{M}_N)$ spanned by the sections F_l.

$2\pi i m h/e^2\sigma_{\mathrm{H}}$ is the trace of the curvature of a rank m vector bundle over $\hat{\Sigma}$ which is homeomorphic to $W_N(\mathcal{L}_m)$. To show that this is so, notice that:

(1) Extending the basis set $\{F_l\}$ of h^0 to a basis of $H^0(\hat{\Sigma}\times\Sigma\times\cdots\times\Sigma, \mathcal{M}_N)$ $\langle\sigma_{\mathrm{H}}\rangle$ essentially computes the slope of the bundle $\pi_{\hat{\Sigma}*}\mathcal{M}_N$. The integration in the particle coordinates z_i meant by the hermitian metric $\langle|\rangle$ is translated in algebro-geometric terms by taking the direct image $\pi_{\hat{\Sigma}*}\mathcal{M}_N$ to the first factor.

(2) The subspace $h^0(\hat{\Sigma}, \pi_{\hat{\Sigma}*}\mathcal{M}_N)$ is isomorphic to the space $H^0(\hat{\Sigma}, W_N(\mathcal{L}_m))$ of holomorphic sections of $W_N(\mathcal{L}_m)$.

Therefore, $\langle\sigma_{\mathrm{H}}\rangle = (e^2/h)\nu_{\mathrm{HR}}$ is also proportional to the slope of $W_N(\mathcal{L}_m)$. Given the isomorphism $W_N(\mathcal{L}_m) \cong W_{N-1}(\mathcal{L}_m)$, it shows that the identity $\nu_{\mathrm{HR}} = |\mu(W_1(\mathcal{L}_m)|$ is not accidental. By establishing these isomorphisms we have shown that $\sigma_{\mathrm{H}} = 1/m$ for all values of N. This is the mathematical statement of the existence of an effective mean field theory given by $W_1(\mathcal{L}_m)$. Moreover, a direct approach on the enormous calculation involved in the determination of the integrand in (46) was made by Varnhagen, using asymptotic methods when $N \to \infty$, in the appendix of [11].

His result

$$\sigma_{\mathrm{H}} \overset{N\to\infty}{\cong} \frac{e^2}{h} \int_0^1 \mathrm{d}\phi_1 \int_0^1 \mathrm{d}\phi_2 \frac{1}{m} = \frac{e^2}{h} |\mu(W_1(\mathcal{L}_m))| \tag{47}$$

fits in perfectly with our isomorphism $W_N(L_m) \cong W_{N-1}(L_m)$ because he finds an N-independent answer.

On the other hand, Varnhagen's computation suggests that, in parallel with the Thouless analysis in the IQHE, $\langle\sigma\rangle = \sigma_{\mathrm{H}}$ also happens in the FQHE. Geometrically, this means again that our bundles $W_N(\mathcal{L}_m)$ are homeomorphic to the F_m bundle characterized by the $U(m)$ connection

$$\omega = -\frac{2\pi \mathrm{i}}{m} I_{m\times m} \phi_1 \mathrm{d}\phi_2$$

$I_{m\times m}$ is the unity $m \times m$ matrix, of constant curvature:

$$\mathrm{tr}\, R_\omega = -2\pi \mathrm{i} \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2.$$

According to Donaldson [20], the indecomposable bundle F_m is stable because the trace of the curvature is constant in $\hat{\Sigma}$. We have by no means shown, however, that $W_N(\mathcal{L}_m)$ and F_m are holomorphically equivalent.

$W_N(\mathcal{L}_m)$ is a point in the moduli space of stable bundles of degree 1 over $\hat{\Sigma}$, identified by Narasimhan and Seshadri [21] as being homeomorphic to $S^1 \times S^1$. There exists a family of vector bundles $E_m(c_1, c_2)$ with unitary connections

$$\frac{1}{2\pi \mathrm{i}} \omega(c_1, c_2) = \left(-\frac{1}{m}\phi_1 + c_2\right) \mathrm{d}\phi_2 + c_1 \mathrm{d}\phi_1$$

$c_1, c_2 \in [0, 1]$, which are the absolute minima of the Yang–Mills equations on $\hat{\Sigma}$,

$$\mathrm{d}^*_\omega R_\omega = 0$$

modulo automorphisms. The curvature for any $\omega(c_1, c_2)$ is

$$R_\omega = -\frac{2\pi \mathrm{i}}{m} I_{m\times m} \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2$$

so that $c_1(E_m(c_1, c_2)) = (1/2\pi \mathrm{i}) \int_{\hat{\Sigma}} \mathrm{tr}\, R_\omega = -(1/m) \cdot m = -1\ \forall c_1, c_2$.

A qualitative explanation of the rôle played by the moduli space of stable bundles in the existence of plateaux is as follows.

Because the FQHE occurs only in high-mobility samples, only weak impurities are of interest. Strong impurities destroy the FQHE and the physical mechanism leading to the

IQHE plateaux does not work. Tao and Haldane [6] study how a potential describing both the impurities and a uniform background affects the Haldane–Rezayi wave functions and the topological nature of the σ_{H} in the FQHE. Adiabatic changes in the solenoid fluxes and impurities produce changes in the filling factor. The estimation of the modification induced in σ_{H} requires one to know the effect of the impurities on the degenerated ground state. Under the assumption that impurities are not strong enough to fill the gap to the first excited state, i.e.

$$V_{\mathrm{imp}} : \Psi_l^{\mathrm{HR}} \longrightarrow \mathrm{e}^{Ec_1\phi_1+\mathrm{i}c_2\phi_2} \sum_{j=1}^{m} a_j \Psi_j^{\mathrm{HR}},$$

where $a_j \in \mathbb{C}$, $c_a \in [0, 2\pi]$ are constant. The non-zero amplitude transitions induced by V_{imp} between the HR wave functions means that the degeneration of the ground state disappears and new eigenstates arise that differ in energy by a very small amount from it

$$\Psi_l^{\epsilon_l} = \mathrm{e}^{c_1\phi_1+\mathrm{i}c_2\phi_2}\Psi_l^{\mathrm{HR}},$$
$$(H_0 + H_{\mathrm{imp}})\Psi_l^{\epsilon_l} = (E_0 + \epsilon_l)\Psi_l^{\epsilon_l}.$$

Since the splitting in energy is so small due to the weakness of the impurities, the Kubo–Thouless formula is still applicable. The important point, not explicitly noticed in [6], is that phases appear in the new lower energy sections as a response to the action of the impurities on the center of mass dynamics in the presence of solenoid fluxes.

The connection obtained from these new sections corresponds to $E_m(c_1, c_2)$, the HR state being the $E_m(0, 0)$ case. Thus, variations in the filling factor merely produce variations in the moduli space of stable bundles; the slope of the bundle being the Hall conductance, σ_{H} remains the same all over the moduli space; this explains the plateaux. This is an amazing result: a highly complex two-dimensional gas of interacting fermions self-organizes itself in such a way that it can be described by a stable bundle, which is a very subtle concept in algebraic geometry. We wonder whether, as in the Yang–Mills system over a Riemann surface, there is any relationship between dynamical stability and algebro geometric stability in this multi-particle fermionic system.

A final comment: The role of localization in the IQHE is crucial as regards the existence of plateaux. Variations in electron density when ϕ_1, ϕ_2 vary as functions of time do not affect σ_{H} because the change in ν_{H} occurs in localized states around impurities that are not conducting. We have seen that in the FQHE the variations due to changes in ϕ_1, ϕ_2 and the switching of weak impurities merely produce “motion” in the moduli space of fiber bundles. There is no role for localization in this argument due to non-binding impurities; the FQHE occurs in samples with very low levels of disorder!.

5. Summary and outlook

In this long paper we have addressed the problem of developing a mathematically sound formulation of the theory of the quantum Hall effect in a periodic lattice. We found algebraic geometry as the natural framework and by completing this program we have achieved the following goals:

1. When restricted to the first Landau level, the fermionic Fock space for a two-dimensional gas of electrons moving on a lattice is identified as the space of holomorphic sections of certain line bundles.
2. Application of the addition formulas of algebraic geometry to the wave functions in the Fermionic Fock space provides a generalization of Slater determinant developments to the periodic case, allowing analytic computation of the filling factor.
3. Quasi-hole and quasi-particle Fock spaces, as well as the corresponding wave functions, are also identified.
4. The non-trivial center of mass dynamics arising in the periodic case is described in terms of the space of sections of a vector bundle. This vector bundle is rigorously defined.
5. It is shown that the Hall conductance is the slope of the above-mentioned bundle. A topological argument for the existence of plateaux is thus provided. Moreover, it is proved that the CM vector bundle is stable.

As a future research in this area we propose two lines of both physical and mathematical interest:

– We plan to prove the existence of canonical hermitian structures over the bundles $W_N(\mathcal{L}_m)$. This would explain the special character of the variational ground state, the Haldane–Rezayi wave function, between the sections in $H^0(\hat{\Sigma}, W_N(\mathcal{L}_m))$. Also, random lattices enter our approach by allowing one to vary the modular parameter. Both modifications, to be performed in a future work, suggest strong theoretical connections between the FQHE and conformal field theories.
– To be precise, the explanation of Section 4 is referred to the first plateaux. To extend the arguments to other experimentally observed values it is necessary to consider the K-matrix theory of Zee and Wen [22], see also [23]. The theory developed in this paper will be generalized to this more general situation by replacing elliptic curves by abelian varieties in a forthcoming paper.

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